Below are the four final questions from the World Math Olympiad Qualifiers.

1. Find the smallest possible sum of two integers whose GCF is 8 and whose LCM is 80.

2. On a list of nine fractions whose average is $\frac{1}{8}$, the first 8 fractions average $\frac{1}{9}$. What is the ninth fraction on the list?

3. How many perfect squares are factors of $8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$?

4. Compute: $\frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 3} - \frac{1}{3 \cdot 4} - \frac{1}{4 \cdot 5}$.

1. For two numbers to have GCF 8, both numbers must include at least three 2's in their prime factorizations.

For the LCM of the pair of numbers to be $80=2^4\cdot 5$, exactly one of the two numbers must have a 5 in its prime factorization. Similarly, exactly one of the two numbers must have four 2's in its prime factorization. (If both had a 5 or four 2's, their GCF would no longer be 8).

Since we want the sum of the two numbers to be as small as possible, neither number should have any additional prime factors.

This gives two possibilities. One number can have the 5 and the four 2's, giving the pair $2^4\cdot 5=80$ and $2^3=8$.

Or, one number has the 5 and the other has the four 2's, giving the pair $2^3\cdot 5=40$ and $2^4=16$.

$40+16=56$ is less than $80+8=88$, so **56** is the smallest possible sum.

2. Nine numbers whose average is $\frac{1}{8}$ have a sum of $9\cdot\frac{1}{8}=\frac{9}{8}$.

The eight fractions whose average is $\frac{1}{9}$ sum to $8\cdot\frac{1}{9}=\frac{8}{9}$.

So, the ninth fraction is $=\frac{9}{8}-\frac{8}{9}=\frac{81}{72}-\frac{64}{72}=\mathbf{\frac{17}{72}}$.

3. We begin with the prime factorization of 8!:

$$\begin{aligned}8! &= 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1\\ &= (2^3)\cdot 7\cdot (2\cdot 3)\cdot 5\cdot (2^2)\cdot 3\cdot 2\\ &= 2^7\cdot 3^2\cdot 5\cdot 7.\end{aligned}$$

The factors in the prime factorization of a perfect square can be divided into two identical groups. So, there will always be an even number of each prime factor in the prime factorization of a perfect square. So, we cannot have any 5's or 7's in the prime factorization of a perfect square factor of (8!).

But, we can have zero, two, four, or six 2's. We can also have zero or two 3's. This gives the perfect squares below. Recall that any number raised to the 0 power is 1.

$2^0\cdot 3^0=1$ $\quad$ $2^0\cdot 3^2=9$

$2^2\cdot 3^0=4$ $\quad$ $2^2\cdot 3^2=36$

$2^4\cdot 3^0=16$ $\quad$ $2^4\cdot 3^2=144$

$2^6\cdot 3^0=64$ $\quad$ $2^6\cdot 3^2=596$

So, **8** perfect squares are factors of 8!.

4. To compute this difference, we can use a common denominator of $2\cdot 3\cdot 4\cdot 5=120$:

$$\begin{aligned}&\frac{1}{1\cdot 2}-\frac{1}{2\cdot 3}-\frac{1}{3\cdot 4}-\frac{1}{4\cdot 5}\\ &=\frac{1}{2}-\frac{1}{6}-\frac{1}{12}-\frac{1}{20}\\ &=\frac{60}{120}-\frac{20}{120}-\frac{10}{120}-\frac{6}{120}\\ &=\frac{24}{120}\\ &=\mathbf{\frac{1}{5}}.\end{aligned}$$

There is a nice pattern in the differences below. To find it, compute each of the following:

$$\frac{1}{1\cdot 2}=$$

$$\frac{1}{1\cdot 2}-\frac{1}{2\cdot 3}=$$

$$\frac{1}{1\cdot 2}-\frac{1}{2\cdot 3}-\frac{1}{3\cdot 4}=$$

$$\frac{1}{1\cdot 2}-\frac{1}{2\cdot 3}-\frac{1}{3\cdot 4}-\frac{1}{4\cdot 5}=$$

$$\frac{1}{1\cdot 2}-\frac{1}{2\cdot 3}-\frac{1}{3\cdot 4}-\frac{1}{4\cdot 5}-\frac{1}{5\cdot 6}=$$