Beast Academy 3
Chapter 1: Shapes

AoPS Beast Academy
Art of Problem Solving

Sequence: BA3, Chapter 1 Shapes ▶ BA4, Chapter 1 This Chapter ▶ BA5, Chapter 1 3D Solids

The Shapes chapter can be difficult for students, particularly if this is their first time using Beast Academy. It can be moved to any point in the BA3 sequence. **It does not need to come first**.

Overview

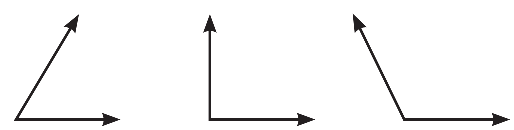
This chapter is primarily about classifying shapes.

We introduce lots of new terms with rigorous mathematical definitions. It is fine for students to use a reference sheet to organize new terms and definitions.

Classifying Angles

Angles are classified by how “open” they are (the size of an angle has nothing to do with the lengths of its “sides”). Lines that meet at a perfect “L” form a **right** angle. An angle that is less open than a right angle is **acute**, and an angle that is more open than a right angle is called **obtuse**.

Angle Classifications



Classifying Triangles

Triangles are classified by both their angles and their sides.

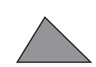
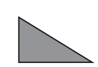
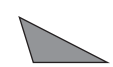
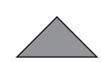
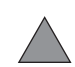
Angles:

If the angles are all acute, the triangle is an **acute** triangle.
A triangle that has a right angle is a **right** triangle.
A triangle that has an obtuse triangle is an **obtuse** triangle.

Sides:

If the sides are all different lengths, the triangle is **scalene**.
If *at least* two sides are equal, the triangle is **isosceles**.
If all three sides are equal, the triangle is **equilateral**.

Triangle Classifications

	Acute	Right	Obtuse
Scalene			
Isosceles			
Equilateral		X	X

Teaching:

Students should not use measuring tools. For angles, it’s enough to compare to the corner of an index card or other right angle. For isosceles triangles, students should look for symmetry. “If you fold the triangle in half, will the sides match?” Students should rotate the triangles to get different views.

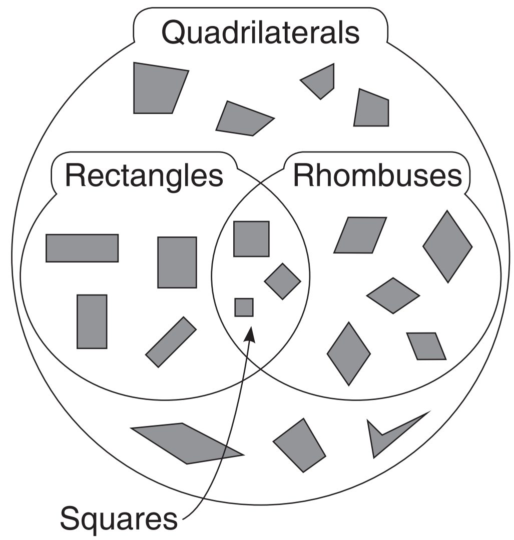
Ask students why there are two “missing” triangle types in the chart above. Every equilateral triangle looks the same (ignoring size), with three acute angles. So, *all* equilateral triangles are acute.

Consider the definition of isosceles triangles. Ask students whether an equilateral triangle is also isosceles. (Yes, since an equilateral triangle has *at least* two equal sides, it is also isosceles.)

Beast Academy 3
Chapter 1: Shapes

Classifying Quadrilaterals

Quadrilateral Classifications



We can use a **Venn diagram** to help students see the relationships between different types of shapes.

Shapes that have straight sides are called **polygons**.

A polygon with four sides is called a **quadrilateral**.

A quadrilateral with four right angles is a **rectangle**.
A quadrilateral with four equal sides is a **rhombus**.

A quadrilateral that is both a rectangle *and* a rhombus (four right angles and four equal sides) is called a **square**.

It may be hard to convince students that this shape () is a rectangle. Every square is a special kind of rectangle (one with four equal sides). Every square is also a special kind of rhombus (one with four right angles).

Give students examples of classification systems they are familiar with. For example, all pigeons are birds, and all birds are animals. In the same way, all squares are rectangles, and all rectangles are quadrilaterals.

Ask classification questions like, “Is every rectangle a quadrilateral?” (Yes.) Students who answer “No” should try to give examples. “Is every rectangle a square?” (No. )

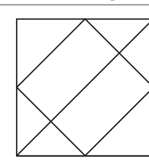
Play with shapes in unexpected orientations. “What words describe this shape: ?” (Rectangle, quadrilateral, polygon, and shape.)

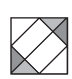
Counting Shapes

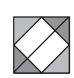
We ask students to count shapes. These can be very difficult for students and require careful organization as described in BA2 Chapter 12.

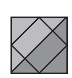
Students also get practice recognizing shapes that are made from smaller shapes. Understanding how shapes can be built from smaller pieces is key to solving many geometry problems.

How many triangles are in this figure?



 There are 4 small triangles,

 4 medium triangles,

 and 2 large triangles.

So, there are a total of $4+4+2=10$ triangles.

Other Shapes and Puzzles

Polyominoes are included in this chapter to give students some fun spatial reasoning practice. The puzzles towards the end of the chapter can take a lot of time and fiddling. A goal of puzzles like these is to give students opportunities to try and fail. Students are usually more comfortable failing at problems with simple manipulatives like polyominoes or toothpicks, where trial-and-error experimentation are necessary and easily encouraged.

An important trait for all good problem solvers is resilience and a willingness to fail.

Beast Academy 3
Chapter 2: Skip-Counting

AoPS Beast Academy
Art of Problem Solving

Sequence: ... BA2, Chapter 8 Strategies (+ & −) ▸ **BA3, Chapter 2 This Chapter** ▸ BA3, Chapter 4 Multiplication ...

Overview

This chapter focuses on skip-counting and patterns that will help students transition to multiplication. Students who practice skip-counting learn to recognize useful patterns and gain number sense (for example, recognizing that certain numbers come up a lot, like 12, 30, and 42, and others never come up, like 19, 23, and 41.)

Before learning multiplication, skip-counting helps students develop strategies for moving quickly between multiples of a number. For example, by recognizing that ten 7's is 70, students can jump to nearby multiples of 7 by skip-counting.

−7 +7 +7 +7 +7

...63, 70, 77, 84, 91, 98, ...

This is also useful for helping students practice multiplication facts. For example, students who know $6\times6=36$ can quickly get to 7×6, 8×6, or 9×6 by completing a known skip-counting pattern.

+6 +6 +6

...36, 42, 48, 54,...

6×6 7×6 8×6 9×6

Patterns

Encourage students to recognize patterns when skip-counting. The hundreds chart can help students see these patterns.

For example, when skip-counting by a number that is close to 10, students can add 10, then compensate. Adding 9 is the same as adding 10 then taking away 1. Adding 12 is the same as adding 10 then adding 2.

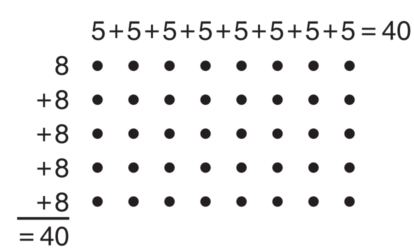
Skip-counting by 12
on the hundreds chart.

11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60

Commutativity

In this chapter, we lay the foundation for the commutative property of multiplication ($5\times8=8\times5$). Students should see that adding five 8's gives the same result as adding eight 5's.

This allows students to quickly add fifty 3's, for example, by adding three 50's.



Beast Academy 3
Chapter 2: Skip-Counting

Starting at Different Numbers

Standard skip-counting by 6's begins with 6 and continues with the multiples of 6: 6, 12, 18, 24, 30, 36, and so on.

We also encourage students to begin at other numbers. For example, students may be asked to count by 6's starting at 180. When they later multiply 33×6 (or 6×33), they can start at 30×6 = 180, then add three more 6's to get 198.

Eventually, they will do this without skip-counting. This repeated-addition model of multiplication helps students see that multiplication is distributive: 6×33 = (6×30) + (6×3).

Add thirty-three 6's.
–or–
Multiply 33×6.

$$\underset{30\times6}{180}, \xrightarrow{+6} 186, \xrightarrow{+6} 192, \xrightarrow{+6} \underset{33\times6}{198}.$$

Balance Scale Problems

This section is optional. It is based on a classic problem, which asks for the largest number of chicken nuggets that *cannot* be ordered if they are sold in boxes of 6, 9, and 20. In other words, what is the largest number you cannot get by adding 6's, 9's, and 20's.

It is an example of a difficult problem that can be solved with organized reasoning and has a surprising result. The table on the right shades all of the numbers you *can* get by adding 5's and 7's. You can get every number past 23!

It is useful to note that this *doesn't work if the numbers you are adding share a common factor*. For example, if both are even (like 4 and 6), it is impossible to get any odd results.

Numbers you can get by adding 5's and 7's.

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	32	33	34	35
36	37	38	39	40	41	42

Beast Academy 3
Chapter 3: Perimeter & Area

Sequence: BA3, Chapter 1 Shapes → BA3, Chapter 3 This Chapter → BA3, Chapter 12 Area

Overview

This chapter introduces perimeter and area as two ways to measure a shape's size, then present a variety of problems to help students build a deep understanding of these concepts.

Units: To keep things simple, we do not include units in this chapter. Feel free to include units if your students are ready, but we don't recommend requiring them in student answers at this level.

Multiplication: Multiplication is not introduced until BA3, Chapter 4, and we purposefully avoid perimeter and area formulas in this chapter. However, it's fine if students multiply to compute areas.

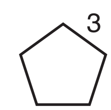
Finding areas helps to motivate multiplication, and gives student an excellent visual model for later multiplication strategies.

Perimeter

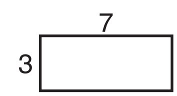
Perimeter is the distance around a shape, and can be found by adding all of a shape's side lengths. Quickly move to problems where not all of the side lengths are labeled.

For example, in a regular polygon (where all sides are equal), knowing just one side length is enough.

In a rectangle, opposite sides are the same length. So, you only need two side lengths to find its perimeter.



Perimeter:
$3+3+3+3+3=15$



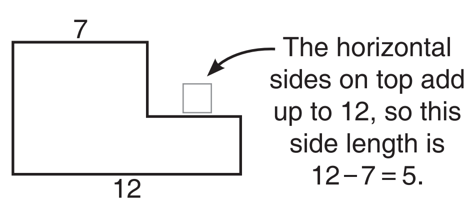
Perimeter:
$3+7+3+7=20$

Rectilinear Shapes

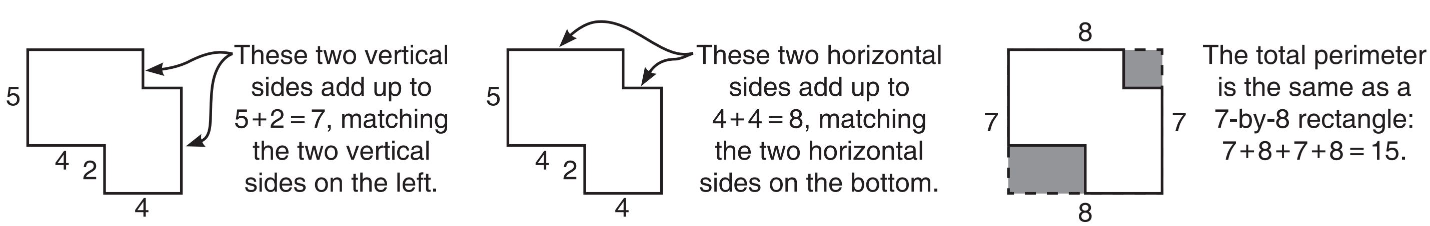
Rectilinear shapes have sides that meet at right angles. Encourage students to use the relationships between sides to find missing side lengths.

Once students are comfortable, challenge students to find the perimeter of a rectilinear shape *without* finding all of its side lengths.

Find the missing side length in the rectilinear shape below.



Find the perimeter of the rectilinear shape below.



Beast Academy 3

Chapter 3: Perimeter & Area

Triangle Inequality

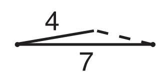
This is an accessible topic that helps students think carefully about the relationships of side lengths in a triangle. Help students discover that in a triangle, the lengths of the two short sides must add up to more than the length of the long side.

Guide students with questions about concrete examples. “Can you attach sticks that are 3, 5, and 9 inches long to make a triangle? Why or why not?”

Once students can recognize which examples are impossible, ask, “If two of the sticks are 4 and 7 inches, what are the possible lengths of the third stick?”

A triangle has sides that are 4 and 7 inches long. What are the possible lengths of its third side?

The third side has to be **less** than $4+7=11$ inches.



The third side has to be **more** than $7-4=3$ inches.

Area

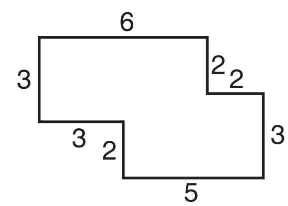
Area is the amount of space a flat shape takes up. We can divide shapes into equal-sized squares to find and compare areas.

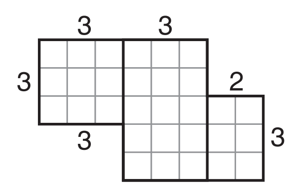
For rectangles, the side lengths give us the number of squares tall and wide a shape is. We can use these to count the number of squares needed to cover the rectangle (by skip-counting, or multiplying for students who have already learned their basic multiplication facts).

Students then learn to split complicated rectilinear shapes into rectangles, adding partial areas to find the total area of a shape.

Breaking shapes into parts is an essential skill in geometry. As students encounter more difficult shapes (triangles, trapezoids, parallelograms, etc.), they will be able to split them into familiar parts to help them recognize properties, find areas, and discover formulas on their own.

What is the area of the rectilinear shape below?





The total area is 30 squares (square units).

Area and Perimeter Relationships

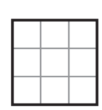
A common misconception is that a shape with more area will always have more perimeter (and vice-versa).

Encourage students to draw shapes with the same perimeter, then compare their areas.

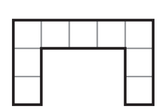
Draw shapes with the same area and compare their perimeters.

Help students discover that more area does not *always* mean more perimeter.

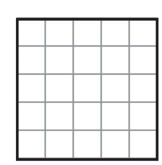
The two shapes on the left have the same area, but very different perimeters.
The two shapes on the right have the same perimeter, but very different areas.



Perimeter: 12
Area: 9 sq



Perimeter: 20
Area: 9 sq



Perimeter: 20
Area: 25 sq

Beast Academy 3
Chapter 4: Multiplication

AoPS Beast Academy
Art of Problem Solving

Sequence: BA3, Chapter 2 Skip-Counting → BA3, Chapter 4 This Chapter → BA3, Chapter 6 The Distributive Property → BA4, Chapter 2 Multiplication

Multiplication can be used as a shortcut for repeated addition. This chapter transitions students from skip-counting to memorizing multiplication facts.

Overview

This is one time when memorization is critical.

Students who can quickly recall every basic multiplication fact (to at least 10×10) will have a significant advantage over those who can't. For example, a student who does not recognize that both 45 and 63 are divisible by 9 will have no idea how to simplify $\frac{45}{63}$.

Students will memorize multiplications facts at different rates. We recommend **including extra drill as needed** to help all students memorize their multiplication facts, and continuing to supplementing with multiplication practice for students who need it.

The Times Table

Memorizing 121 multiplication facts may seem daunting. Help students recognize that there isn't much to memorize.

Start with a basic times table (0×0 through 10×10). This can be filled by **skip-counting**. Students should have had significant practice with skip-counting.

Review the chart with your students. Students should notice some helpful **patterns**. Practice multiplying by 0, 1, 2, and 10. We will learn some quick ways to multiply by 4 and 5 later.

Help students see that almost every fact has a **"twin"** (3×7 = 7×3). If you know one, you know the other.

If we remove the ×0, ×1, ×2, and ×10 facts, and one of each pair of "twin" facts (like 3×7 and 7×3), there are just 28 facts to memorize.

×	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10
2	0	2	4	6	8	10	12	14	16	18	20
3	0	3	6	9	12	15	18	21	24	27	30
4	0	4	8	12	16	20	24	28	32	36	40
5	0	5	10	15	20	25	30	35	40	45	50
6	0	6	12	18	24	30	36	42	48	54	60
7	0	7	14	21	28	35	42	49	56	63	70
8	0	8	16	24	32	40	48	56	64	72	80
9	0	9	18	27	36	45	54	63	72	81	90
10	0	10	20	30	40	50	60	70	80	90	100

Memorizing **perfect squares** is useful. If you know 7×7 = 49, it is easy to get to 8×7 and 9×7 by counting up by 7's. Almost all of the hardest facts to memorize are within 1 or 2 steps of a perfect square.

With **practice**, students should eventually move away from skip-counting methods to recalling facts.

Memorizing multiplication facts early will provide huge benefits. Students will use them for the rest of their life, both in school and outside the classroom.

Memorizing perfect squares will help students quickly compute near-perfect-squares by counting up.

×	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10
2	0	2	4	6	8	10	12	14	16	18	20
3	0	3	6	9	12	15	18	21	24	27	30
4	0	4	8	12	16	20	24	28	32	36	40
5	0	5	10	15	20	25	30	35	40	45	50
6	0	6	12	18	24	30	36	42	48	54	60
7	0	7	14	21	28	35	42	49	56	63	70
8	0	8	16	24	32	40	48	56	64	72	80
9	0	9	18	27	36	45	54	63	72	81	90
10	0	10	20	30	40	50	60	70	80	90	100

Beast Academy 3
Chapter 4: Multiplication

Multiples of 10

In our number system, each place value is ten times the place value to its right. So, multiplying a number by 10 shifts each digit one place to the left. In other words, multiplying a whole number by 10 puts a 0 at the end of the number. For example, $3\times10=30$, and $222\times10=2{,}220$.

Multiplying a number by 10 adds a zero to the end of the number. Multiplying by more than one 10 adds more zeros.

Since $100=10\times10$, multiplying by 100 is the same as multiplying by 10 twice. So, multiplying a whole number by 100 puts two 0's at the end of the number.

$$\begin{aligned}&222\times100\\=\,&222\times10\times10\\=\,&22{,}200\end{aligned}$$

We can use the same logic to multiply two numbers that have zeros at the end.

$$\begin{aligned}&500\times3{,}000\\=\,&(5\times10\times10)\times(3\times10\times10\times10)\\=\,&(5\times3)\times10\times10\times10\times10\times10\\=\,&1{,}500{,}000\end{aligned}$$

With practice, students will learn to ignore the zeros at the end of numbers, multiply, then write the correct number of zeros at the end of the product.

Careful, though. The number of zeros in the numbers you are multiplying doesn't always match the number of zeros in their product, as shown in the 50×600 example on the right.

$$\begin{aligned}&50\times600\\=\,&(5\times10)\times(6\times10\times10)\\=\,&(5\times6)\times10\times10\times10\\=\,&30{,}000\end{aligned}$$

Strategies

Reordering
The Commutative and Associative properties of multiplication let us reorder and group the numbers in a multiplication expression however we want. A good strategy is pairing numbers that make 10's or multiples of 10.

$$\begin{aligned}&2\times7\times5\times3\\=\,&(7\times3)\times(2\times5)\\=\,&21\times10\\=\,&210\end{aligned}$$

Multiplying by 4 (and 8)
Many students have an easy time doubling numbers. Since $4=2\times2$, to multiply by 4, students can double twice. For example, $13\times4=13\times2\times2$. Doubling 13 gives 26, and doubling 26 gives 52.
Similarly, since $8=2\times2\times2$, to multiply a number by 8, students can double the number three times.

$$\begin{aligned}&13\times4\\=\,&13\times2\times2\\=\,&26\times2\\=\,&52\end{aligned}$$

Multiplying by 5
Students have not been introduced to division yet, but can still often find half of a number. Since 5 is half of 10, multiplying by 5 gives a result that is half as much as multiplying by 10.

28×5 is half of $28\times10=280$.
Half of 280 is 140.
So, $28\times5=140$.

Combining Strategies
Students can combine strategies and make up their own. For example, to multiply by 50, students can multiply by 100 and find half. Encourage students to explain why their strategies work.

62×50 is half of $62\times100=6{,}200$.
Half of 6,200 is 3,100.
So, $62\times50=3{,}100$.

Beast Academy 3
Chapter 5: Perfect Squares

Sequence: BA3, Chapter 3 Perimeter & Area → BA3, Chapter 4 Multiplication → BA3, Chapter 5 This Chapter

Overview

This chapter gives students an opportunity to practice multiplication. We place an emphasis on area models that give students some clever computation strategies. Perfect squares help students improve their number sense and estimation skills, and offer a foothold towards working with exponents.

We do not use any exponent notation (x^2) in this chapter.

Basics

Squaring a number means multiplying the number by itself.

We call the result of squaring a whole number a **perfect square**. For example, 0, 25, and 81 are all perfect squares.

Memorizing the first 10 to 15 perfect squares will help students later with other computations and estimation.

After 0, what are the five smallest perfect squares?

The five perfect squares after 0 are $1\times1=\underline{1}$, $2\times2=\underline{4}$, $3\times3=\underline{9}$, $4\times4=\underline{16}$, and $5\times5=\underline{25}$.

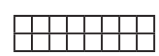
Area and Perimeter

We focus on an area model of multiplication. Students relate squaring a number to finding the area of a square.

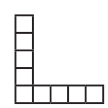
Students can compare the areas of rectilinear shapes (shapes that have only right angles) that have the same perimeter.

Students should discover that for a given perimeter, a square is the rectilinear shape with the largest area.

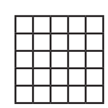
Which shape below with perimeter 20 has the largest area?



Perimeter = 20
Area = <u>16</u>



Perimeter = 20
Area = <u>9</u>



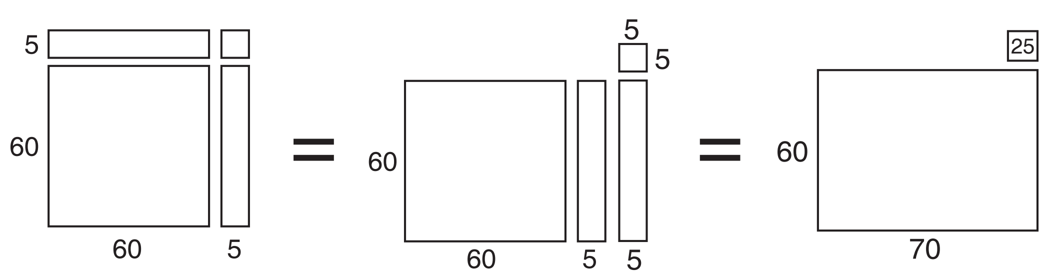
Perimeter = 20
Area = <u>25</u> ✓

Visual Models of Computation Strategies

Squares Ending in 5

Encourage students to understand the math behind any “trick” you present. In this case, we use an area model to explain a quick way to square any number that ends in 5. For example, to square 65, we can multiply 60×70 and add 25, as shown by the area model below.

We can use the same reasoning to square 45 (which is 40×50+25), or 95 (90×100+25), or any other number ending in 5.

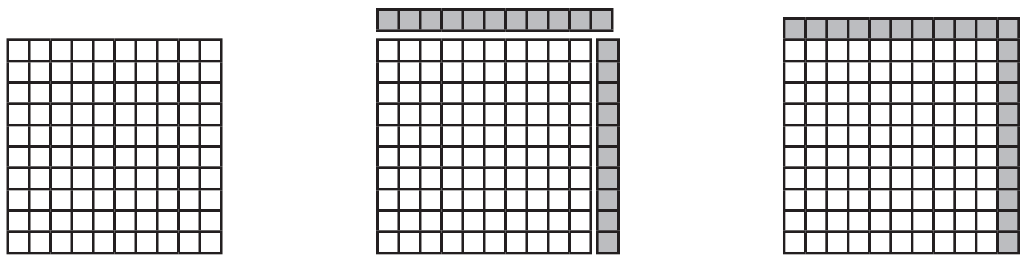


Visual Models of Computation Strategies (cont.)

Finding Nearby Perfect Squares

We can get from a known perfect square to a nearby perfect square by adding or subtracting.

Compute 11×11.



We know that $10\times10=100$.

To get to 11×11, we can add $10+11$ more unit squares.

$$\begin{aligned} & 11\times11 \\ = & (10\times10)+10+11 \\ = & 121 \end{aligned}$$

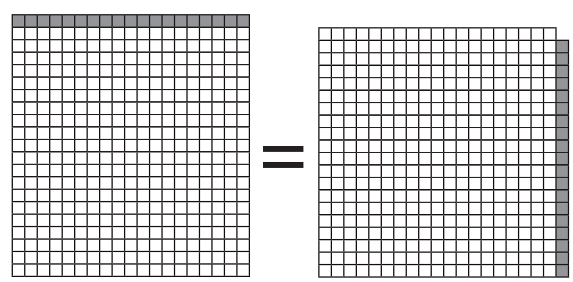
Multiplying Nearby Numbers

There is a useful relationship between the product of numbers that are two apart (like 4×6) and the square of the number between them (5×5).

Playing with some easy examples (like 8×10 vs 9×9), students should see a pattern. The product of the two numbers is always 1 less than the square of the number between them. For example, 19×21 is one less than 20×20.

A visual model can help students understand why this is always true. If we start with a 19-by-21 square, we can rearrange the unit squares to make a 20-by-20 square with 1 unit missing, as shown on the right.

Compute 19×21.



$$\begin{aligned} 19\times21 &= (20\times20)-1 \\ &= 399 \end{aligned}$$

In Algebra, students will encounter special products like $(x-1)(x+1)=x^2-1$ and can use the same visual model to understand them.

Extras

Perfect squares are fun and show up all over the place. In this chapter, we include a few cute topics and curiosities.

- The sum of the first n odd numbers equals n^2 (discussed by R&G in the Guide).
- Every whole number can be written as the sum of four or fewer perfect squares (Practice pg 65).
- We explore dissecting a square into the smallest number of smaller squares possible, sometimes called a Mrs. Perkins' Quilt problem (discussed in the Lab section of the Guide).

These problems are easy for young students to explore and give fun opportunities for enrichment.

Beast Academy 3
Chapter 6: The Distributive Property

Sequence: BA3, Chapter 3 Perimeter & Area → BA3, Chapter 4 Multiplication → **BA3, Chapter 6 This Chapter** → BA4, Chapter 2 Multiplication

Overview

The distributive property is the foundation of every common multiplication algorithm. It allows students to split multiplication into parts, and then add those parts to get a final product.

Many students apply it intuitively to problems like, “How many days are there in 52 weeks?” Students find ways to break the multiplication into parts that are easy to multiply. “In 50 weeks there are $50\times7=350$ days, so two more weeks gives us $350+14=364$ days.” Students learn to write this mathematically as $7\times(50+2)=(7\times50)+(7\times2)$.

Later, in algebra, students will use the distributive property to expand expressions like $x(x+7)$ to get x^2+7x. Students who have used the distributive property for multi-digit multiplication will have a much easier time understanding how to apply it with more abstract expressions in algebra.

Order of Operations

The order of operations tells us how to evaluate an expression so that everyone gets the same result.

To evaluate an expression, start with what’s in parentheses. Multiplication is done before addition and subtraction. Add and subtract from left to right.

The distributive property gives us a way to rewrite an expression that includes parentheses without changing its value. For example, $(60+7)\times3$ can be rewritten as $60\times3+7\times3$.

Evaluate $60\times3+7\times3$.

We don’t just work from left to right. We multiply before we add.

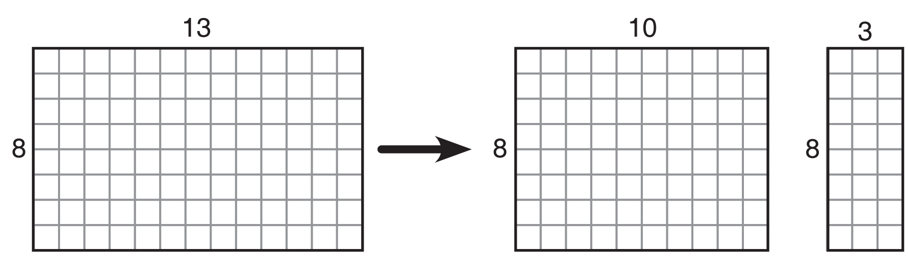
$$\begin{aligned} &60\times3+7\times3\\ =\ &180+21\\ =\ &201\end{aligned}$$

The Area Model

Area models give students a great way to visualize multiplication and the distributive property.

When finding the area of a rectangle, it is often easiest to split the rectangle into parts with sides that are easy to multiply. The total area of the rectangle is the sum of its parts.

Find the area of an 8-by-13 rectangle.



$$\begin{aligned} 8\times13 &= 8\times10 + 8\times3\\ &= 80 + 24\\ &= 104 \text{ (square units)}\end{aligned}$$

Students may find other ways to split the rectangle to get the same answer.

Beast Academy 3
Chapter 6: The Distributive Property

Distributing

Encourage students to mentally compute products of 1-digit numbers times 2- or 3-digit numbers. For example, to multiply 106×4, guide students to think about how many 4's they are adding.

106 fours is the same as 100 fours plus 6 fours. So, students can add 100×4 plus 6×4 to get $400+24=424$. Students then learn to show their work by applying the distributive property more formally. Guide them initially with arrows and blanks to fill in. With practice, students should be able to write the work on their own.

$$106\times4=\left(\boxed{100}+\boxed{6}\right)\times\boxed{4}=\boxed{100}\times\boxed{4}+\boxed{6}\times\boxed{4}=\boxed{400}+\boxed{24}=\boxed{424}.$$

Expressions should include multiplication both left and right of the parentheses.

$$6\times(20+3)=6\times20+6\times3=120+18=138.$$

Students should also learn to distribute multiplication over subtraction. $(40-3)\times7$ can be rewritten as $40\times7-3\times7$. Again, this makes sense. $(40-3)$ sevens is the same as 40 sevens minus 3 sevens.

$$(40-3)\times7=40\times7-3\times7=280-21=259.$$

Factoring

Students should also use the distributive property "in reverse" to rewrite expressions. This is called factoring.

For example, to compute $19\times7+31\times7$, students can combine these products into one. Ask, "How many 7's are there all together?"

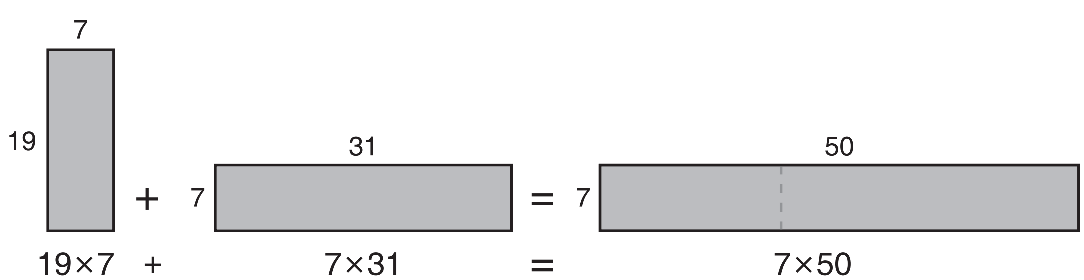
"19 sevens plus 31 sevens is 50 sevens.
So, $19\times7+31\times7$ is $50\times7=350$."

Explain this in words (as above), using the distributive property (on the right), and using area models (below).

Evaluate $19\times7+31\times7$.

Since both of the products we're adding include a 7, we can factor a 7, then compute:

$$\begin{aligned}&19\times7+31\times7\\=\;&(19+31)\times7\\=\;&50\times7\\=\;&350\end{aligned}$$



Students should gain fluency changing the order of two terms that are added or multiplied. For example, students should recognize that all of the expressions below are equal.

$$\underline{(19+31)}\times\underline{7}=7\times(19+31)\qquad 31\times7+\underline{7}\times\underline{19}=31\times7+19\times7\qquad \underline{7\times31}+\underline{7\times19}=7\times19+7\times31$$

Beast Academy 3
Chapter 7: Variables

Sequence: BA2, Chapter 5 Expressions → BA3, Chapter 7 This Chapter → BA5, Chapter 3 Expressions & Equations

We recommend that students complete Chapter 5 in BA2 before beginning this chapter. It introduces students to expressions, using symbols like ⬣, ▲, and ☾, as variables.

Overview

This chapter introduces students to the traditional use of variables (letters that stand for numbers). Introducing variables early reduces the intimidation factor that comes if students see them for the first time in prealgebra and helps students explain and generalize relationships and patterns.

Basics

A variable is a symbol (usually a letter) that stands for a number.

Students have used symbols like ⬣, ▲, and ☾, or empty boxes that stand for numbers (as in $16+\square=58$). Encourage students to think of letters the same way they think of shapes or empty boxes.

Figuring out what x is in $16+x=58$ is no different than filling in the empty box in $16+\square=58$, or figuring out what ⬣ stands for in 16+⬣=58.

Solving for x is similar to problems students have done before.

$16+\square=58$

16+⬣=58

$16+x=58$

Expressions

An **expression** uses numbers, variables, and math symbols (like $+$, $-$, ( ), and later $\times$, $\div$, and more) to stand for a value. For example, 14, $a-b$, and $2\times n$ are examples of expressions. Expressions don't have equals signs (equations do).

Evaluating an expression means finding its value. If an expression includes a variable, the value of an expression can change, depending on the value of the variable.

Evaluate $16+x$ if $x=20$.

$16+x$
$=16+20$
$=36$

Simplifying an expression means writing it in a way that makes it easier to use without changing its value. For example, students can see that adding a number and then subtracting the same number is the same as doing nothing.

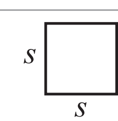
Students can see that $14+n-n$ can be simplified to 14, and $x+79-79$ can be simplified to x. Similarly, $x+15-16$ is simpler as $x-1$.

Simplify $x+79-79$.

$x+\cancel{79-79}$
$=x$

We also explore how expressions can be used to describe patterns and create formulas. For example, the perimeter of a square with side length s is always $4\times s$, and its area is always $s\times s$.

Write an expression for the area and perimeter of a square with side length s.



Perimeter: $4\times s$
Area: $s\times s$

Beast Academy 3
Chapter 7: Variables

Equations

An **equation** is a mathematical statement in which two expressions are equal. Put simply, **equations have equals signs**. For example, $14+6=20$, $a-b=7$, and $16=2\times n$ are all equations.

Solving Equations (Guess-and-Check)

Solving an equation means finding the value of the variable. We encourage students to solve equations like $16+x=58$ the same way they solve $16+\boxed{}=58$. Students can see that $16+\underline{42}$ is 58, so x must stand for 42.

Sometimes, it helps to **simplify** part of an equation before solving it.

Students who can **guess and check** should be allowed to. **Don't discourage guessing and checking.**

Solve for x if $16+x+11=50$.

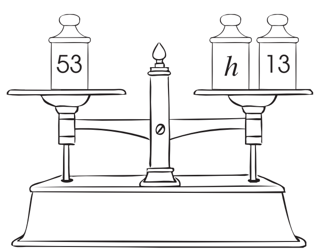
First, $16+x+11$ is the same as $x+27$. So, $x+27=50$.

Since $\underline{23}+27=50$, $x=23$.

Solving Equations ("Balance Scale" Methods)

Whatever is done on one side of a balance must also be done on the other for the two sides to stay equal. Similarly, subtracting or adding the same amount on both sides of an equation can help you find the value of a variable.

We can remove 13 grams from both sides of the balance scale on the left, and the h-gram weight will balance the remaining $53-13=40$ grams on the left side.



Similarly, we can subtract 13 from both sides of the equation on the right, leaving us with the equation $40=h$.

$$\begin{array}{rcl} 53 &=& h+13 \\ -13 & & -13 \\ \hline 40 &=& h \end{array}$$

Balance scale strategies can be helpful when equations are difficult to guess-and-check. Presenting this as an advanced strategy used in algebra may help motivate bright students to learn another useful method for solving more complex equations.

Translating

Translating words into math is not necessary at this level, but helps prepare students for the more complicated problems they will solve using expressions and equations later on.

Encourage students to use meaningful variables (s for the number of cookies <u>S</u>am made, for example) and always test that their expressions and equations match the words.

Tom made 5 more cookies than Sam. Together, they made 33 cookies. How many cookies did Tom make?

If Sam made s cookies,
then Tom made $s+5$ cookies.
Together, they made $s+(s+5)$ cookies. So,

$$s+(s+5)=33.$$

Solving for s, we get $s=14$. So, Sam made 14 cookies and Tom made $14+5=19$ cookies.

Beast Academy 3
Chapter 8: Division

Sequence: BA3, Chapter 4 Multiplication → BA3, Chapter 6 Distributive Property → BA3, Chapter 8 This Chapter → BA4, Chapter 2 Multiplication •••

Students must have a solid understanding of multiplication and must know their basic multiplication facts from BA3, Chapter 4, including products of multiples of 10.

Overview

Division is splitting an amount into equal parts. We explore several ways to think about division.

Most importantly, students must understand the relationship between division and multiplication. To compute $63 \div 7$ efficiently, a student must know that 9×7 is 63.

Since 9 groups of 7 make 63, splitting 63 items into groups of 7 makes 9 groups.

A student who doesn't know 9×7 is 63 will have a tough time with $63 \div 7$.

Understanding multiplication is critical to understanding division.

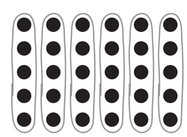
Basics

Division is used to answer two types of questions:

How many groups?

If we know the total number of items and we know how many items we want in each group, we can figure out how many equal groups there will be.

30 students are split into groups of 5.
How many groups are there?

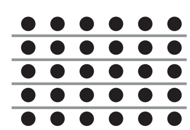


We can divide 30 into 6 groups of 5.
So, $30 \div 5 = 6$.

How many are in each group?

If we know the total number of items and we know how many equal groups there will be, we can figure out how many items are in each group.

30 students are split into 5 equal groups.
How many are in each group?



We can divide 30 into 5 equal groups of 6.
So, $30 \div 5 = 6$.

Students who understand both models above can think about division problems whichever way is easiest for them.

For example, to solve $90 \div 18$, it's easier to imagine 90 split into groups of 18 than splitting 90 into 18 groups. A student reasons, "I can add 18's until I get 90. It takes five 18's to make 90, so $90 \div 18 = 5$."

To solve $96 \div 3$, it's easier to imagine 96 items split into 3 equal groups than counting how many 3's are in 96. A student may reason, "To split 96 into 3 equal groups, I first put 30 in each group. That leaves 6 more to split up. I can put 2 more in each group to get 3 groups of 32. So, $96 \div 3 = 32$."

Relating Multiplication and Division

Multiplication is taking a number of equal amounts and finding a total. For example, 3 groups with 5 in each group makes 15 all together: $3 \times 5 = 15$.

Division is splitting a total into equal amounts. For example, dividing 15 objects into groups of 5 makes 3 groups: $15 \div 5 = 3$. Or, dividing 15 objects into 3 equal groups makes groups of 5: $15 \div 3 = 5$.

Beast Academy 3
Chapter 8: Division

Relating Multiplication and Division (continued)

We can use any known multiplication fact to help us divide. For example, knowing $4\times7=28$ can help us solve two division problems as shown below.

4 groups of 7 makes 28: $4\times7=28$.

So, 28 divided into 4 groups makes 7 in each group: $28\div4=7$.

And 28 divided into groups of 7 makes 4 groups: $28\div7=4$.

It can be useful to think of division as **finding a missing number in a product**. This is especially true when computing with multiples of 10.

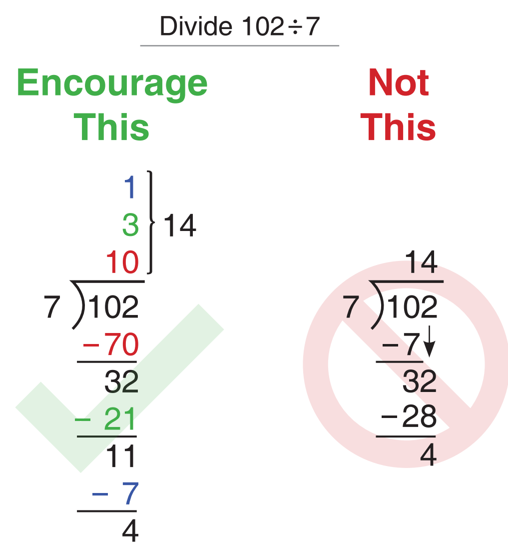
To solve $20{,}000\div40$, we can find the number that fills the blank in $40\times\square=20{,}000$. Since $40\times500=20{,}000$, we know that $20{,}000\div40=500$.

Long Division and Remainders

Before teaching any long division algorithms, help students reason their way through some division problems with larger numbers using concrete examples.

"If I divide 102 apples equally into 7 equal baskets, how many will be in each basket? Will there be any extra apples? How many?"

Encourage students to discuss their strategies. For example, "We can put at least 10 apples in each basket. That's 70 apples, so we'll have $102-70=32$ left. Then, we can put 3 more in each basket. That leaves $32-21=11$ apples. Then, if we put 1 more apple in each basket, there will be $11-7=4$ apples left. We can't put another whole apple in each basket. So, each basket gets $10+3+1=14$ apples, with 4 left over."



Both methods give $102\div7=14$ with 4 left over.

The method on the right looks shorter, ***but*** it's unclear what happened.

The method on the right may take more writing, but it relates to an intuitive process that students understand and can talk through.

The long division algorithm we encourage is just a way for students to organize this thought process. Note that using 10, 3, and 1 is one of *many* ways students can complete the division with this algorithm. Its flexibility is another advantage our algorithm has over the traditional algorithm.

We don't suggest teaching the standard long division algorithm on the right. It turns an intuitive process into a set of mysterious steps that are difficult to explain.

"Why does this work? Why do we divide 7 into 10, not 102? Why do we 'bring down' the 2? What is going on!? This doesn't make any sense!"

Many students become frustrated with math around the time long division is introduced. Teaching an algorithm that students can't make sense of causes math to lose meaning. We urge you to teach an algorithm that helps build conceptual understanding.

Please read the Guide and the BA4 Chapter 5 overview for more.

Beast Academy 3
Chapter 9: Measurement

Sequence: BA2, Chapter 7 Measurement → BA3, Chapter 9 This Chapter

This chapter can be used at any time with students who are comfortable with multiplication and division and can even be sprinkled throughout BA level 3. There are even some short projects that can be used independently to give students hands-on practice with measurement.

Overview

In this chapter, the Guide book and practice problems do not correspond directly. The Guide covers broad measurement concepts and is organized by measurement system (Customary and Metric). The practice is organized by measurement type (length, weight, volume, temperature, price, and time.) We recommend reading the entire chapter in the Guide first before beginning the practice.

Measurements give us ways to describe physical properties with numbers. **Units** make these measurements meaningful. Measurements without units are not useful.

"How long is it?"   "It's 5!"   "Five what? Inches? Miles? Hours?"

The goal in this chapter is to give students a broad overview of the various types of measurements, measurement systems, and the units within these systems. Students learn important practical skills such as how to measure, estimate, and make basic computations and conversions.

Units

A unit is a specific amount that can be used to describe a physical property like length, weight, or temperature.

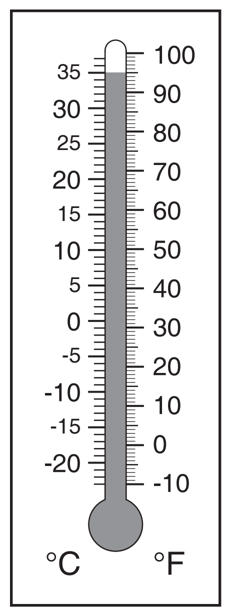
Measurement Systems

Beast Academy includes units from two different systems of measurement: the **Customary** and **Metric** systems (apologies to our overseas readers who undoubtedly find customary units absurd).

Each system has its own sets of standardized units that allow people around the world to measure and describe things accurately.

Students do not need to memorize conversion factors, but should have a general sense of the size of common units. For example, a gram is about the weight of a paperclip, and a kilogram is about the weight of a textbook. Students should also be able to choose an appropriate unit when measuring.

Generally, big units are used to measure big things (the drive is best described as 19 kilometers, not 1,900,000 centimeters), and small units are used to measure small things (the pencil is best described as 19 cm long, not 0.00019 kilometers long).



35° Celsius (95° Fahrenheit) is hot, no matter where you are in the world.

Addition and Subtraction

To add or subtract two measurements, they must use the same units. To add 3 hours to 120 minutes, we can't simply add $3+120$. We need to convert the measurements into the same units, as described in the next section. ($180+120=300$ minutes, or $3+2=5$ hours.)

Beast Academy 3
Chapter 9: Measurement

Unit Conversions

We can convert between units in each measurement system (we do not convert between units in different systems at this level). For example, below are some common units of volume and capacity and their conversions. Again, students should not be expected to memorize these.

Units of Volume and Capacity

Customary Unit	Conversion	Metric Unit	Conversion
fluid ounce (fl oz)		milliliter (mL)	
cup (c)	1 c = 8 fl oz	liter (L)	1 L = 1,000 mL
pint (pt)	1 pt = 2 c		
quart (qt)	1 qt = 2 pt		
gallon (gal)	1 gal = 4 qt		

Large to Small

Converting a large unit into a smaller one like cups into fluid ounces is similar to reasoning students have used before. If there are 8 apples in each basket, 3 baskets hold $3 \times 8 = 24$ apples. Similarly, since there are 8 fluid ounces in each cup, 3 cups equals $3 \times 8 = 24$ fluid ounces.

Small to Large

Converting from a small unit to a large one is often more difficult for students, and it helps to start with familiar units like seconds and minutes (or feet and inches for U.S. students). Students can use a variety of strategies to answer, “How many minutes are in 180 seconds?”

Students may make a table, count up, divide, or think of this as regrouping seconds into minutes. “Since there are 60 seconds in 1 minute, 120 seconds is 2 minutes, and 180 seconds is 3 minutes.”

Mixed Measures

Some measurements are commonly given with two different units. For example, lengths in feet and inches, weights in pounds and ounces, or time in hours and minutes. Students can convert from a small unit to a mixed measure. For example, students can write 30 inches as 2 ft 6 in, or 200 minutes as 3 hrs 20 mins. (The number of small units must always be too small to regroup into a large unit. For example, we never write 2 hrs 80 mins.)

How many ounces are in 3 pints?

We can use the table above.

1 pint is 2 cups, so 3 pints is 6 cups.

1 cup is 8 ounces, so 6 cups is 48 fl oz.

3 pints = 48 fl oz.

Write 50 ounces in pounds and ounces.

1 pound is 16 oz.

2 pounds is 32 oz.

3 pounds is 48 oz.

So, 50 oz is 2 oz more than 3 pounds.

50 oz = 3 lb 2 oz.

Types of Measurements

The problems in the Practice book include measurements of length, weight, volume, temperature, price, and time. These can be done in any order.

Several projects are included in the chapter that give students opportunities for hands-on measurement, experimentation, and practical applications.

Beast Academy 3
Chapter 10: Fractions

AoPS Beast Academy
Art of Problem Solving

Sequence: BA3, Chapter 8 Division → **BA3, Chapter 10 This Chapter** → BA4, Chapter 8 Fractions → BA4, Chapter 2 Fractions …

Students must have a solid understanding of multiplication and division before beginning this chapter.

Overview

This chapter introduces fractions. Establish early that **fractions are numbers on the number line.**

Fractions are often introduced as parts of a whole. This can lead to confusion. For example, thinking of $\frac{3}{7}$ as "three out of seven" makes it hard to see why $\frac{3}{7}+\frac{3}{7}$ equals $\frac{6}{7}$. In the context of free throws, three out of seven plus three out of seven is six out of fourteen!

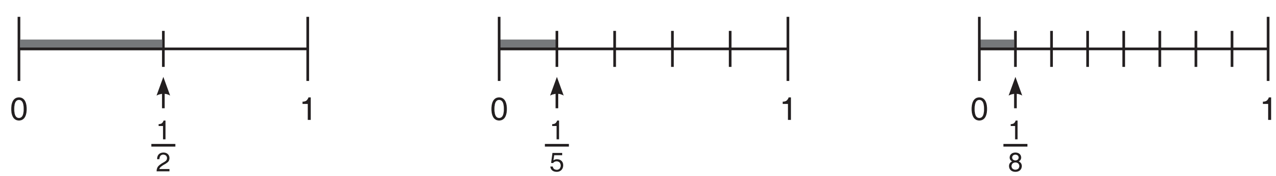
Using the model of fractions as numbers on the number line helps students connect them to the numbers that they already understand.

Fractions on the Number Line

Fractions are another way to write division. For example, $1\div 2$ can be written as $\frac{1}{2}$.

The number being divided is called the numerator. → $\frac{1}{2}$ ← The number you are dividing by is called the denominator.

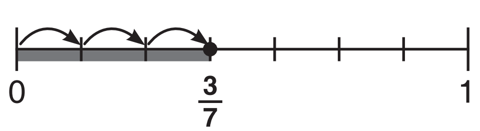
If we divide the number line between 0 and 1 into 2 pieces of equal length, each piece has a length of $\frac{1}{2}$. Similarly, if we divide the number line between 0 and 1 into 5 pieces of equal length, each piece has length $\frac{1}{5}$. If we divide it into 8 pieces, each piece has length $\frac{1}{8}$.

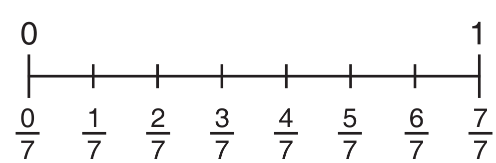


Fractions with 1's in their numerators are called **unit fractions**.

The more pieces you split something into, the smaller those pieces are. So, the larger the denominator, the smaller the unit fraction. For example, $\frac{1}{9}$ is a little bit smaller than $\frac{1}{8}$.

We can build other fractions from unit fractions just by counting up on the number line. For example, we can start at 0 and count 3 pieces of length $\frac{1}{7}$ to find $\frac{3}{7}$. We can mark all of the sevenths from 0 to 1 as shown on the right. Make sure students are counting pieces, not tick marks.





It makes sense that $\frac{7}{7}=1$, since $\frac{7}{7}$ means the same thing as $7\div 7$!

Fractions can be greater than 1. For example, $\frac{28}{7}$ equals $28\div 7=4$.

Other fractions are between whole numbers. For example, $\frac{16}{7}$ is a fraction that is a little more than $\frac{14}{7}$ (which equals 2), but less than $\frac{21}{7}$ (which equals 3).

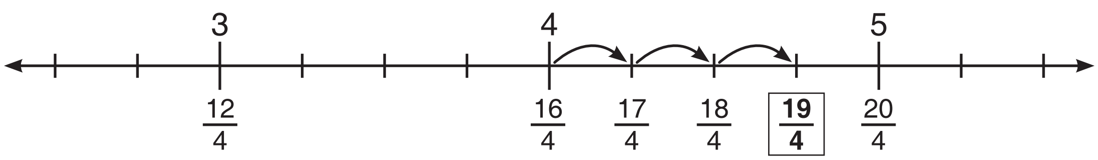
Beast Academy 3
Chapter 10: Fractions

Mixed Numbers

A **mixed number** is a way to write a fraction that is greater than 1. For example, $3\frac{1}{6}$ is a mixed number that means $3+\frac{1}{6}$.

We can use whole numbers to help us write fractions that are greater than 1 as mixed numbers. For example, we can compare $\frac{19}{4}$ to some nearby whole-numbers.

$\frac{19}{4}$ is between $\frac{16}{4}=4$ and $\frac{20}{4}=5$. We can label $\frac{19}{4}$ on the number line as shown below.

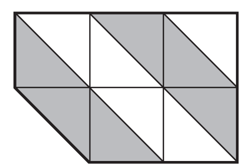


So, $\frac{19}{4}$ is $4+\frac{3}{4}$. As a mixed number, we write $\frac{19}{4}$ as $4\frac{3}{4}$. Avoid algorithms with conversion "tricks." Read the BA4 Chapter 8 overview for more on converting between fractions and mixed numbers.

Parts of a Whole

Fractions can be used to represent a part of any whole. We recommend saving this approach until students fully understand fractions as numbers on the number line.

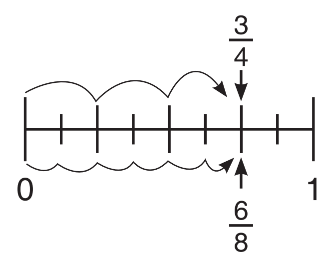
Have students practice shading fractions of shapes by splitting them into equal parts (given by the denominator) and shading some of the parts (given by the numerator).



$\frac{6}{11}$ of the shape is shaded.

Equivalent Fractions

Fractions that stand for the same point on the number line are called **equivalent fractions**. Fractions that are equaivalent look different, but are equal. For example, $\frac{3}{4}$ and $\frac{6}{8}$ are equivalent.



If we take three fourths and split each fourth into two pieces, we get six eighths. This is the same as multiplying both the numerator and denominator by 2. Multiplying or dividing the numerator and denominator of a fraction by the same number converts it into an equivalent fraction.

When we divide, it is called **simplifying** the fraction. A fraction is in **simplest form** when 1 is the only whole number that divides both the numerator and denominator with no remainder. Two fractions are equivalent (and therefore equal) if they have the same simplest form.

$$\frac{3}{4} \overset{\times 2}{=} \frac{6}{8} \overset{\times 10}{=} \frac{60}{80} \overset{\div 4}{=} \frac{15}{20} \overset{\div 5}{=} \frac{3}{4}$$

Comparing and Ordering

We can compare two fractions that have the same numerator or denominator.

$\frac{6}{7} > \frac{5}{7}$ *Six* sevenths is more than *five* sevenths because there are more sevenths.

$\frac{5}{8} < \frac{5}{7}$ Five *eighths* is less than five *sevenths* because eighths are smaller than sevenths.

Sometimes, you need to convert one or both fractions to compare them.

Beast Academy 3
Chapter 11: Estimation

AoPS Beast Academy
Art of Problem Solving

Sequence: ••• BA3, Chapter 4 Multiplication ▶ BA3, Chapter 8 Division ▶ BA3, Chapter 10 Fractions ▶ BA3, Chapter 11 This Chapter •••

Students must have a solid understanding of multiplication, division, and fractions before beginning.

Overview

Estimation is a practical skill that helps build number sense. We estimate when we don't need an exact answer (How many apples are in an orchard?), to make a prediction (How long will the drive take?), when an exact answer is impossible (How many planets are there in the galaxy?), or to make sure an amount is reasonable (Did I pay the right amount?). An estimate is a thoughtful guess.

Some students will be reluctant to estimate when they can find an exact answer. This is one area when time constraints can help emphasize the value of estimation skills.

Rounding

For rounding to a place value, emphasize a common-sense approach.

Round 6,971 to the nearest hundred.

Encourage This

6,971 is between 6,900 and 7,000.

Since 6,971 is closer to 7,000,

6,971 rounds to 7,000.

Not This

The hundreds digit is 9. To the right of the 9 is a 7. When the digit to the right is 5 or more, we round up. But, since there is not a digit that is bigger than 9, we turn the 9 into a 0, and the thousands digit goes up by 1 to 7.

The digits to the right of the hundreds place become 0's.

6,971 rounds to 7,000.

Numbers that are exactly in the middle round up. For example, 7,550 rounds to 7,600 when rounded to the nearest hundred, and 8,500 rounds to 9,000 when rounded to the nearest thousand.

Fractions work the same way. $3\frac{5}{11}$ rounds down to 3, while both $3\frac{5}{10}$ and $3\frac{5}{9}$ round up to 4.

Goals of Estimation

There are no strict rules for estimating computations. Don't simply round. There are two main goals:

1. The estimate should be much easier to compute than the exact value.

2. The estimate should be reasonably close to the exact value.

Any estimate that meets both goals is good. When estimating, most or all of the math should be mental, and the amount the estimate is off by should be small compared to the actual value.

Estimate 378×590.

$380\times590=224{,}200$	$400\times600=240{,}000$	$378\times1{,}000=378{,}000$
✗ Easy	✓ Easy	✓ Easy
✓ Close	✓ Close	✗ Close

Only $400\times700=280{,}000$ meets both requirements of a good estimate. 378×590 is actually 223,020.

Beast Academy 3
Chapter 11: Estimation

Computing Estimates

There are usually many good ways to estimate. Encourage a variety of methods as long as they meet the goals of being easy to compute and reasonably close to the actual value.

Be careful when rounding.

To estimate 6×172, a student might round 6 up to 10 and compute $10\times170=1{,}700$. Rounding 6 to 10 in a multiplication problem almost doubles the product! The actual value of 6×172 is 1,032. Help students recognize that rounding 6 up by 4 to 10 gives a bad estimate, since 4 is nearly as big as 6.

Instead, we can use $6\times200=1{,}200$ to get a much better estimate. Rounding 172 up by 28 to 200 still gives a good estimate, since 28 is small compared to 178.

Similarly, to divide $172\div6$, a student might try $170\div6$ or $200\div6$ (which are hard to divide), or $170\div10$ (which gives a bad estimate, since 10 is almost double 6). One good estimate would be $180\div6=30$. In this case, changing 172 to 180 makes the division easy.

Especially when estimating products and quotients, the amount you change each number by should be small compared to the number. Find numbers that are close to the original numbers and easy to compute with. Be flexible. Sometimes it's better not to round.

Over- and Under-estimating

It can be helpful to know whether an estimate is more or less than the actual value. Maybe you're buying paint and you want to make sure you have enough, or loading a cart that has a weight limit you don't want to exceed.

Addition and Multiplication

When estimating a sum or product, if both numbers round up, we get an **overestimate**. For example, estimating $17\times88\approx20\times90=1{,}800$ gives an overestimate. (The $\approx$ means "is approximately.")
For addition and multiplication, if both numbers round down, we get an **underestimate**.

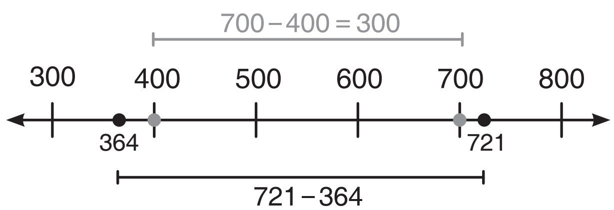
If one number rounds up, and the other rounds down, it's not easy to tell whether the estimate is more or less than the actual value.

Subtraction

When estimating subtraction, it helps to think of the distance between numbers on the number line. If rounding the numbers brings them closer together, we get an underestimate. If rounding the numbers makes them farther apart, we get an overestimate.

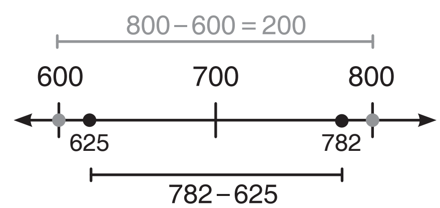
Is $721-364$ more or less than $700-400=300$?

721 and 364 are farther apart than 700 and 400.
So, $721-364$ is **more** than $700-400=300$.



Is $782-625$ more or less than $800-600=200$?

782 and 625 are closer than 800 and 600.
So, $782-625$ is **less** than $800-600=200$.



Beast Academy 3
Chapter 12: Area

Sequence: ••• BA3, Chapter 3 Perimeter & Area ▸ BA3, Chapter 9 Measurement ▸ **BA3, Chapter 12 This Chapter** ▸ BA4, Chapter 1 Shapes •••

Students should have a basic understanding of area from BA3 Chapter 3 and a solid understanding of multiplication before beginning this chapter.

Overview

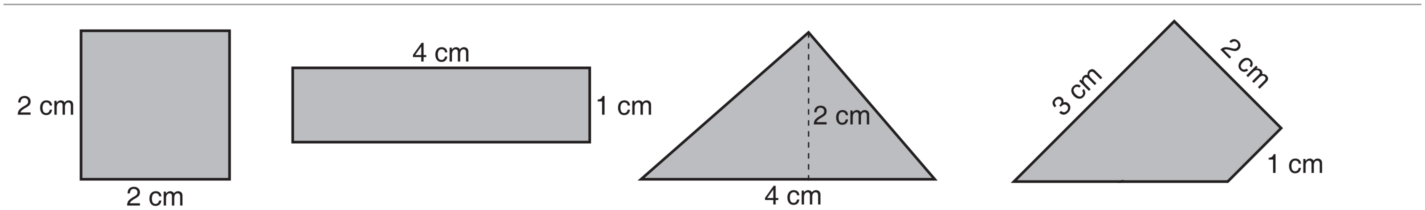
In this chapter, students learn to find the areas of triangles and compound shapes (shapes made by piecing together shapes whose areas they can find).

Avoid teaching area as a set of formulas for students to memorize and apply. Students who learn formulas without understanding them will have trouble remembering them and will have difficulty finding areas of more complex shapes where a formula is unavailable. Instead, make sure students understand the concepts behind these formulas, or, better yet, come up with them on their own.

Measuring Area

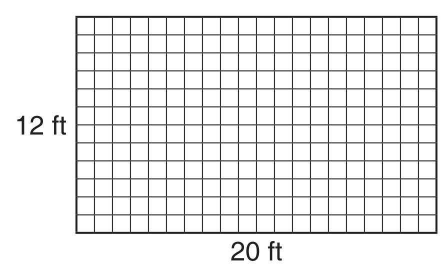
Area is the amount of space a shape takes up on a flat surface. Area is measured in square units. For example, a square centimeter is the area of a square with sides that are 1 cm long. So, if we say that a shape has an area of 4 square centimeters, it takes up the same space as four 1-by-1 cm squares.

All five shapes below area actual size and each has an area of 4 square centimeters.
(This chapter only covers finding the area of the first three shapes.)



Rectangles

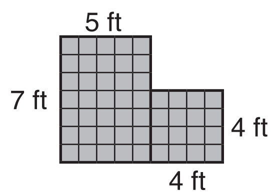
It's easy to split a rectangle with whole-number side lengths into square units. For example, a rectangle that is 20 feet wide and 12 feet tall covers $20\times12=240$ square feet. We can find the area of any rectangle this way.



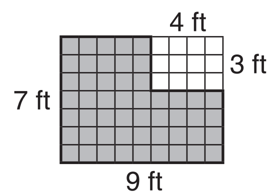
Area
$20\times12=240$ sq ft

Rectilinear Shapes

We can also find the areas of many rectilinear shapes (shapes that only have right angles) by splitting them into rectangles and adding or subtracting rectangle areas.



Addition:
Shaded Area
$=(5\times7)+(4\times4)$
$=51$ sq ft

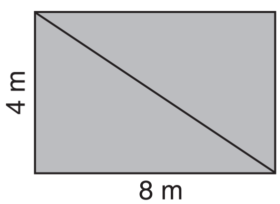


Subtraction:
Shaded Area
$=(9\times7)-(4\times3)$
$=51$ sq ft

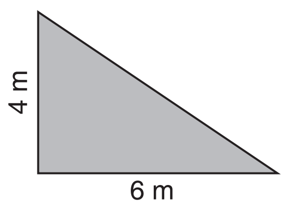
Beast Academy 3
Chapter 12: Area

Triangles

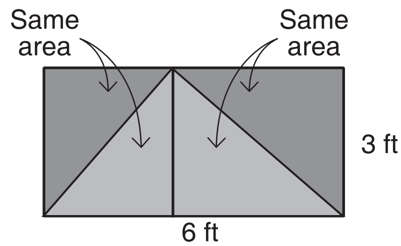
We can arrange two copies of any **right triangle** to make a rectangle. So, every right triangle is half the area of a rectangle. If a triangle is not right, we can still copy it and cut it into right triangles that can be arranged to make a rectangle that has twice its area, as shown below.



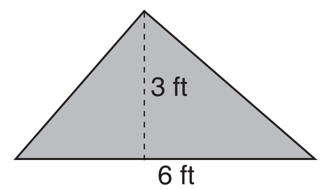
Area = $4 \times 8 = 32$ sq m



Area = $(4 \times 8) \div 2 = 16$ sq m

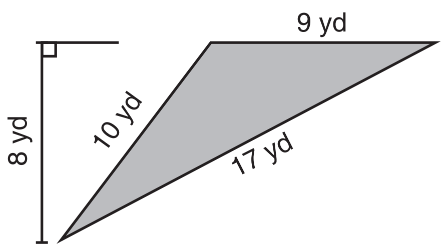


Rectangle Area
$6 \times 3 = 18$ sq ft

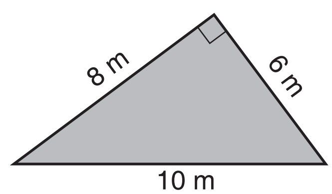


Triangle Area
$(6 \times 3) \div 2 = 9$ sq ft

So, to find the area of any triangle, we multiply the length of its **base** by its height and divide by 2. The base isn't always the side on the bottom. The base is the side of the triangle we measure its height from. The height is always measured at a right angle to the base. Help students recognize which side is the base when finding the areas of triangles that are rotated.



The height is measured from the side that is 9 yards. So, the base is 9 yards and the height is 8 yards. The area is $(9 \times 8) \div 2 = 36$ sq yds.

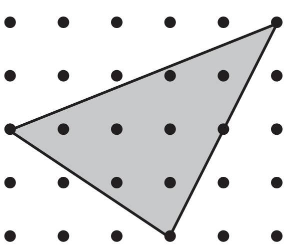


We can use 8 meters as the base and 6 meters as the height, or 6 meters as the base and 8 meters as the height. Either way, we get $(6 \times 8) \div 2 = 24$ sq m.

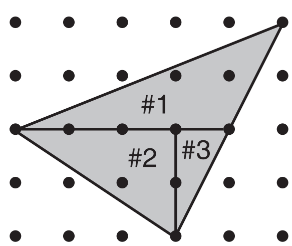
Adding and Subtracting Areas

Many shapes can be split into triangles and rectangles that can then be used to find their areas. Problems like these help ensure students can do more than apply formulas to basic shapes.

Find the area of the shaded triangle in square units.



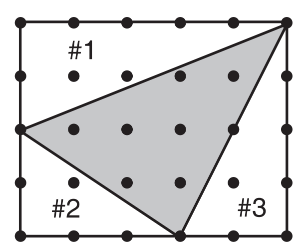
We can find the area of three triangles and add them.



#1: $(4 \times 2) \div 2 = 4$ sq units
#2: $(3 \times 2) \div 2 = 3$ sq units
#3: $(1 \times 2) \div 2 = 1$ sq unit

Shaded Area = $4 + 3 + 1 = 8$ sq units.

Or, we can find the area of the surrounding rectangle and subtract the areas of the three white triangles.



Rectangle: $4 \times 5 = 20$ sq units
#1: $(5 \times 2) \div 2 = 5$ sq units
#2: $(3 \times 2) \div 2 = 3$ sq units
#3: $(2 \times 4) \div 2 = 4$ sq units

Shaded Area = $20 - 5 - 3 - 4 = 8$ sq units.