Beast Academy 4
Chapter 1: Shapes

AoPS Beast Academy
Art of Problem Solving

Sequence: BA3, Chapter 1 Shapes ▸ BA4, Chapter 1 This Chapter ▸ BA5, Chapter 1 3D Solids

This chapter is part of the Geometry sequence. Before starting, students should have a solid foundation with basic shape classification and naming conventions learned in BA3, Chapter 1.

The Shapes chapter can be difficult for students, particularly if this is their first time using Beast Academy. It can be moved to any point in the BA4 sequence. **It does not need to come first**.

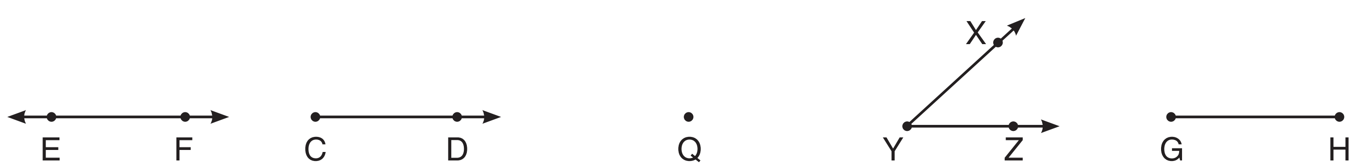
Overview

The same way that understanding numbers and place value is critical to students' early work in arithmetic, understanding the fundamental building blocks of geometry (point, line, segment, etc.), helps students learn how these shapes can be combined into more complex figures later on.

Students learn to measure angles, add new terms to their vocabularies (with clear, rigorous, mathematical definitions), and are introduced to the concept of symmetry.

Definitions

Students should be able to recognize basic geometric figures like line, point, plane, segment, ray, etc., and understand geometric terms. They do not need to memorize these definitions and we recommend students be allowed to reference them as needed.

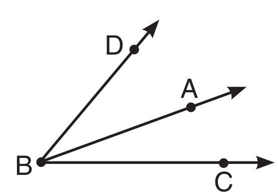


Line EF (or FE) | Ray CD (not DC) | Point Q | Angle XYZ (or ZYX) | Segment GH (or HG)

Angle Measurement

Using a protractor is difficult. While we'd like students to be able to measure angles accurately, it's more important that students gain "angle sense." Encourage students to estimate angle measures within 10 or 15 degrees using benchmark angles like 90°, 45°, and 180°. This will help students recognize which of the two sets of numbers on the protractor to use when measuring.

Students should also recognize that angle measures are additive. For example, in the diagram, there are three angles (DBA, ABC, and DBC). Students should be able to use the measure of any two of these angles to find the third using addition or subtraction.



Understanding how figures can be built from or broken down into smaller pieces is how many geometry problems are solved.

Beast Academy 4
Chapter 1: Shapes

Classifying Shapes (an important note about trapezoids)

Trapezoids require some explanation, and lead to an important point about classification.

Many curricula use what we believe is a flawed definition of trapezoid. They define trapezoid as a quadrilateral with *exactly* one pair of parallel sides. **The widely accepted mathematical definition defines a trapezoid as having *at least* one pair of parallel sides. This is the definition we use.**

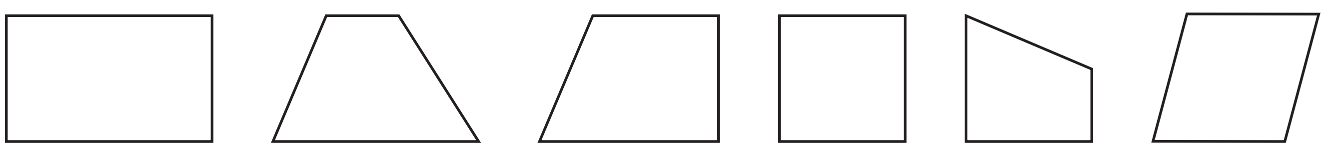
This definition is consistent with all of the other definitions we've used.

For example, a rectangle is a quadrilateral with four right angles. Since a square has four right angles, a square is a special type of rectangle (but it's still a rectangle). A rectangle does not stop being a rectangle just because it has four *equal* sides.

Similarly, a trapezoid should not stop being a trapezoid simply because it has *two* pairs of parallel sides. By our definition, every parallelogram is a special type of trapezoid.

Good classifications describe a category without making "holes" by excluding special cases.

With our definition, the shapes below are all trapezoids (since they are all quadrilaterals with at least one pair of parallel sides). Several have more descriptive name (like parallelogram, rhombus, rectangle, or square), but they are all still trapezoids.



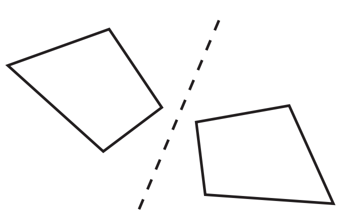
Unfortunately, since there is not universal agreement on the definition of trapezoid, you may need to consult your local standards to make sure your students don't outsmart their tests.

Symmetry

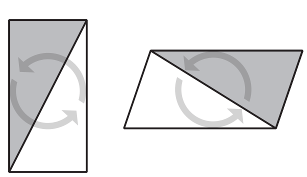
Recognizing symmetry takes practice. In this section, students gain agility with mental rotations and reflections and build pattern recognition when looking at geometric figures.

Being able to recognize symmetries will help students understand relationships like the ones below.

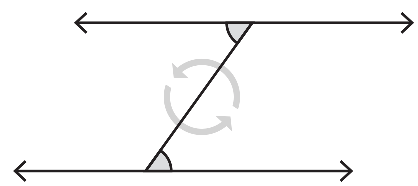
Symmetry can help students recognize congruent shapes.



Rotational symmetry helps students see the 1/2 relationship between triangle and quadrilateral area.



When a line crosses parallel lines, we get pairs of equal angles, which we see with rotational symmetry.



Beast Academy 4
Chapter 2: Multiplication

Sequence: BA3, Chapter 2 Skip-Counting ▸ BA3, Chapter 4 Multiplication ▸ BA3, Chapter 6 The Distributive Property ▸ BA4, Chapter 2 This Chapter

Students beginning this chapter must be fluent with their 1-digit multiplication facts, and should be able to quickly compute products ending in zeros (e.g., $90\times200=18{,}000$ and $50\times80=4{,}000$).

Students should also have a solid understanding of the distributive property (even if they don't know what it's called). We have a whole chapter on the Distributive Property in BA3, Chapter 6.

Overview

Teaching multiplication as a series of steps, of "carrying ones" and "bringing down zeros" doesn't help students understand multiplication. Students rarely know *why* performing these steps works.

We don't recommend teaching any algorithm that students can use without being able to explain why they are doing each step. Students who learn steps without meaning have trouble remembering them, and can't tell when they've made mistakes, since nothing they are doing makes any sense to them.

Instead, focus on helping students *understand* multiplication using the models explained below.

The Distributive Property

The distributive property is the foundation of all the multiplication models we use.

Guide students to see that three 456's is three 400's plus three 50's plus three 6's. That's the distributive property!

Compute 3×456.

$$\begin{aligned}3\times456 &= 3\times(400+50+6)\\ &= (3\times400)+(3\times50)+(3\times6)\\ &= 1{,}200+150+18\\ &= \mathbf{1{,}368}\end{aligned}$$

The Area Model

The area model of multiplication works great as an organizational tool and is a great visual representation of the distributive property.

We are breaking multi-digit multiplication into partial products, then adding them to give a final product.

The area model is especially helpful for introducing 2-by-2-digit multiplication (see below), where the work requires more organization and it's harder to keep track of the partial products.

Compute 3×456.

	400	50	6
3	1,200	150	18

$$\begin{array}{r}1{,}200\\150\\+\ 18\\\hline 1{,}368\end{array}$$

So, 3×456 = **1,368**.

Compute 16×28.

	20	8
10	200	80
6	120	48

$$\begin{array}{r}200\\120\\80\\+\ 48\\\hline 448\end{array}$$

So, 16×28 = **448**.

$$\begin{aligned}16\times28 &= (10+6)\times(20+8)\\ &= (10\times20)+(10\times8)+(6\times20)+(6\times8)\\ &= 200+120+80+48\\ &= \mathbf{448}\end{aligned}$$

The Partial Products Algorithm

Once students understand how to split multi-digit multiplication into partial products, they can learn to organize their work efficiently.

The method we recommend requires more writing than the traditional stacking algorithm shown below, but **students can understand what's happening in each step, and why it works.**

Compute 56×78.

Encourage This

Stack the two numbers, lining up the digits.
Distributing gives four partial products: 8×6 = 48, 8×50 = 400, 70×6 = 420, and 70×50 = 3,500.
Finally, add the partial products.

56	56	56	56	56	56
× 78	× 78	× 78	× 78	× 78	× 78
	48	48	48	48	48
		400	400	400	400
			420	420	420
				3,500	+ 3,500
					4,368

Not This

Stack the two numbers, lining up the digits.
Multiply 8×6 = 48. Write the 8, "carry" the 4. Multiply 8×5 = 40. Add the 4 to get 44 and write that. Bring down a zero.
Multiply 7×6 = 42. Write the 2, "carry" the 4. Multiply 7×5 = 35. Add the 4 to get 39 and write that.
Finally, add the two products.

				4	**4**	4
	4	**4**	4	4	4	4
56	56	56	56	56	56	56
× 78	× 78	× 78	× 78	× 78	× 78	× 78
	8	**44**8	48	448	448	448
			0	**2**0	**3,9**20	+ 3,920
						4,368

Teaching Algorithms

Teaching an algorithm that doesn't promote understanding can do far more harm than good. Students who are taught to view math as a set of processes and formulas often shut down when asked to solve an unfamiliar problem.

Our goal is to help students understand math in a way that helps them become resilient problem solvers who can apply what they've learned to a variety of problems.

Even the recommended algorithm above should only be used when necessary. For example, encourage students to compute 8×103 mentally as 800+24 = 824, rather than stacking the numbers, writing out the partial products, then finding the sum.

Beast Academy 4
Chapter 3: Exponents

AoPS Beast Academy
Art of Problem Solving

Sequence: • • • BA4, Chapter 2 Multiplication ▶ **BA4, Chapter 3 This Chapter** ▶ BA5, Chapter 3 Expressions & Equations ▶ BA5, Chapter 12 Exponents

Students should be fluent with multiplication, perfect squares, and the order of operations (using parentheses, multiplication & division, and addition & subtraction) before beginning this chapter.

Overview

Exponents are a shortcut for writing repeated multiplication. Students should begin by writing out expressions that include exponents as repeated multiplication.

Encourage students to reason their way through problems using what they know about exponents. "For $3^4 \times 3^7$, I am multiplying a total of $4+7=11$ threes. So, $3^4 \times 3^7 = 3^{11}$."

Avoid teaching students to memorize and apply formulas like $a^m \cdot a^n = a^{m+n}$. Students who can reason through problems are likely to discover these formulas on their own and apply them correctly.

Basics

We can write the product $2\times2\times2\times2\times2$ as 2^5. The expression 2^5 is called a **power** of 2 and is read "2 to the 5th power."

In the expression 2^5, the 2 is the **base**. The base is the number we multiply repeatedly. The 5 is called the **exponent**. The exponent tells us how many of the base we multiply.

Practice writing repeated multiplication using exponents ($3\times3\times3\times3=3^4$), and powers as multiplication ($7^3=7\times7\times7$).

Have students evaluate powers using multiplication ($5^4=5\times5\times5\times5=625$) and evaluate nearby powers using a known power as shown on the right. Students who understand exponents will see that 2^9 is $2^8\times2$.

Base → 2^5 ← Exponent

$2^5 = 2\times2\times2\times2\times2$

If $2^8=256$, what is 2^9?

$$\begin{aligned} 2^9 &= (2\times2\times2\times2\times2\times2\times2\times2)\times2 \\ &= (2^8) \times 2 \\ &= 256 \times 2 \\ &= 512 \end{aligned}$$

Students can also multiply powers with the same base. Start with powers that have small exponents where students can write out all of the multiplication. $5^4\times5^3=(5\times5\times5\times5)\times(5\times5\times5)=5^7$.

Then, they can reason through problems with larger exponents without writing out the multiplication.

Write $11^{12}\times11^{13}$ as a power of 11.

Encourage This

$$11^{12}\times11^{13} = \underbrace{(11\times11\times\cdots\times11)}_{\text{12 elevens}}\times\underbrace{(11\times11\times\cdots\times11)}_{\text{13 elevens}}$$

Multiplying $11^{12}\times11^{13}$ gives a product of $12+13=25$ elevens. So, $11^{12}\times11^{13}=11^{25}$.

Not This

Use the formula $a^m \cdot a^n = a^{m+n}$.

When we multiply two powers that have the same base, we add their exponents.

So, $11^{12}\times11^{13}=11^{12+13}=11^{25}$.

Beast Academy 4
Chapter 3: Exponents

Order of Operations

The order of operations tells us how to evaluate an expression. Students should have already practiced with expressions that have parentheses, multiplication & division (working from left-to-right), and addition & subtraction (also done in order from left to right).

Exponents should be evaluated before multiplication and division in the order of operations.

We avoid mnemonics like "PEMDAS." Too many students assume that this means multiplication comes before division and addition comes before subtraction.

Manipulating Expressions with Exponents

Perfect Squares

Have students try to write powers as perfect squares. Students should start by writing powers with small exponents as multiplication, then splitting the numbers into two equal groups. With practice, they can reason their way through working with powers that have larger exponents.

This will help students recognize perfect squares later using a number's prime factorization.

Write 7^4 as a perfect square.

$$\begin{aligned} 7^4 &= 7\times7\times7\times7 \\ &= (7\times7)\times(7\times7) \\ &= 49\times49 \\ &= 49^2 \end{aligned}$$

Equivalent Expressions

Students can practice writing other expressions that are equal (for example, converting 49^2 back to 7^4).

The goal is for students to practice manipulating expressions that include exponents. For example, students should recognize that $(3\times5)^4$ is the product of four 3's and four 5's, which can be written as $3^4\times5^4$.

Later, they will use many of the same strategies for factoring and working with more complex expressions.

Write $2^6\times3^3$ as a power of 12.

$$\begin{aligned} 2^6\times3^3 &= (2\times2\times2\times2\times2\times2)\times(3\times3\times3) \\ &= (2\times2\times3)\times(2\times2\times3)\times(2\times2\times3) \\ &= 12\times12\times12 \\ &= 12^3 \end{aligned}$$

Binary

In our standard base-10 number system, we have ten digits (0-9). The place values are powers of ten: ones (10^0), tens (10^1), hundreds (10^2), thousands (10^3), etc.

Binary (base-2) numbers use just two digits (0 and 1). The place values for binary numbers are powers of 2: ones (2^0), twos (2^1), fours (2^2), eights (2^3), etc.

This is a topic you are unlikely to find in any other curriculum. It's a lot of fun, and learning place value in another system is a great way for students to gain a deep understanding of how our standard base-10 numbers work.

Write thirteen in binary (base-2).

The five smallest place values in base-2 are shown below. Since thirteen is less than sixteen, it is a four-digit number in binary. To make thirteen, we need 1 eight, 1 four, and 1 one.

Sixteens	Eights		Fours	Twos	Ones
___	1	,	1	0	1

So, in binary, thirteen is 1,101.

Beast Academy 4
Chapter 4: Counting

Sequence: BA4, Chapter 4 This Chapter → BA4, Chapter 12 Probability

Counting is a topic in discrete math that is rarely taught at this level. However, it's a lot of fun and requires careful thought and lots of problem solving strategies, so we encourage everyone with time to tackle the math in this chapter. Many of these strategies are used in other parts of the BA curriculum.

Overview

Counting is a lot trickier than the name implies. In this chapter, we introduce skills that are valuable in many areas of math, ranging from probability to statistics to computer science.

More importantly, counting is a fun and approachable topic that requires careful thought, organization, experimentation, and other problem solving strategies.

Counting Lists

How many pages will you read if you start on page 1 and finish on page 10? Most students will correctly answer "10".

How many pages will you read if you start on page 30 and finish on page 50? Many students will wrongly subtract $50 - 30 = 20$. (The correct answer is 21.)

Off-by-1 errors are among the most common mistakes in math, and this section helps students recognize potential errors.

The goal in this section is to help students turn lists like the second into lists like the first (that start at 1 and count by 1's) using addition, subtraction, and division. Changing the numbers in a list does not change how many numbers there are, but it makes the list easy to count.

How many pages will you read if you start on page 30 and end on page 50?

$$\begin{array}{rrrcrrr} 30, & 31, & 32, & \ldots & 48, & 49, & 50. \\ -29 & -29 & -29 & & -29 & -29 & -29 \\ \hline 1, & 2, & 3, & \ldots & 19, & 20, & 21. \end{array}$$

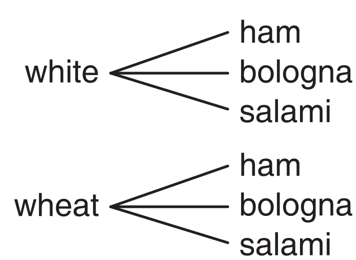
Subtracting 29 from every page number on the list from 30-50 gives us a list from 1-21, which has 21 numbers.

Possibilities (Tree Diagrams)

Counting possibilities requires careful organization. Start with problems that are easy to complete with an organized list before introducing students to tree diagrams, where each branch represents a possibility.

Encourage students to create their own tree diagrams for different problem types before using multiplication to count possibilities.

Sandwiches come on white or wheat bread with a choice of one meat (ham, salami, or bologna). How many possible sandwiches are available?



There are 6 possible sandwiches: ham on white, bologna on white, salami on white, ham on wheat, bologna on wheat, and salami on wheat.

Beast Academy 4
Chapter 4: Counting

Possibilities (Multiplication)

After working with tree diagrams to count possibilities, encourage students to look for shortcuts. “What if there were 4 choices of meat?” “What if there were 14 choices of meat?”

Many will notice that the number of choices at each step can be multiplied to give the total number of possibilities. For example, in our sandwich problem, if there were 5 bread choices, 7 meat choices, and 3 available sizes, this would give $5 \times 7 \times 3 = 105$ possible sandwiches (which would be a real pain to draw the tree diagram for).

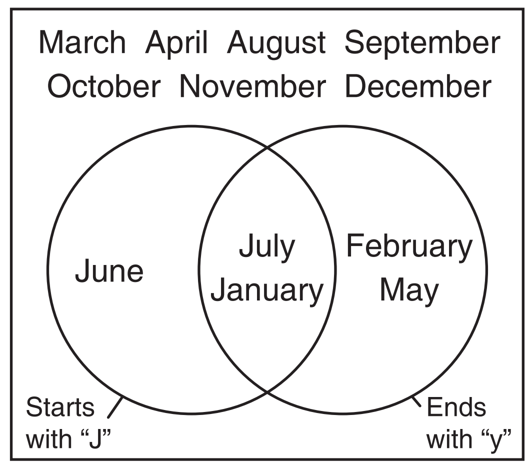
Venn Diagrams

Venn diagrams are useful for organizing things that are in overlapping categories.

Have students learn how Venn diagrams work by helping them create their own. For example, draw overlapping circles to represent students who have black hair and students who have glasses (choose whatever categories work best). Students can write names of classmates in the correct part of the diagram.

Help students recognize that items can be counted in multiple categories. For example, in the diagram on the right, there are 3 months that start with “J”, 4 that end in “y”, 2 that do both, and 7 that don’t do either (but there are not $3+4+2+7=16$ months). Some items get counted more than once.

Months



Arrangements

Students who understand how to count possibilities using multiplication can count arrangements using similar logic.

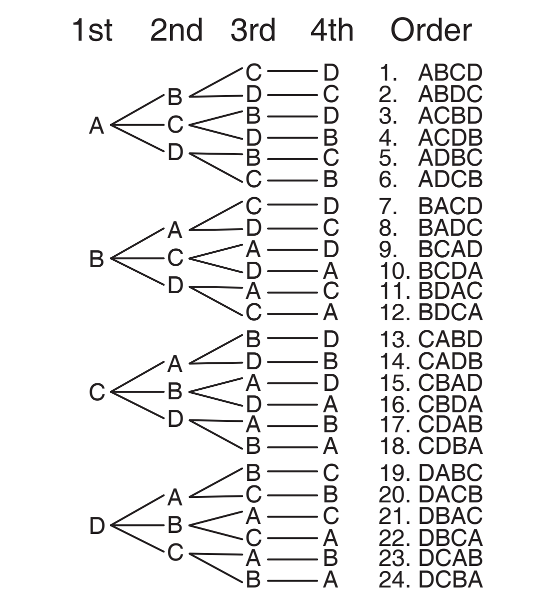
For example, we can count the number of ways Amy, Ben, Cole, and Denise can place 1st through 4th in a race using multiplication (eliminating the need for the painful tree diagram on the left).

Careful! There are not $4 \times 4 \times 4 \times 4$ possibilities.

There are 4 students who can finish first.
This leaves 3 students who can finish second.
This leaves 2 students who can finish third.
The 1 remaining student is last.
So, there are $4 \times 3 \times 2 \times 1 = 24$ orders for the students to finish the race.

The short way to write $4 \times 3 \times 2 \times 1$ is 4!, which is called a **factorial**.

How many ways can Amy, Ben, Cole, and Denise place 1st through 4th in a race?



Beast Academy 4
Chapter 5: Division

AoPS Beast Academy
Art of Problem Solving

Sequence: BA3, Chapter 8 Division → BA4, Chapter 2 Multiplication → **BA4, Chapter 5 This Chapter** → BA4, Chapter 7 Factors

Before beginning this chapter, students should understand the relationship between multiplication and division, and be comfortable finding quotients and remainders as taught in BA3, Chapter 8.

Overview

Students extend the division skills learned in BA3 to larger numbers and learn some new division strategies and divisibility tests.

Special Quotients

Lead a discussion to help students make generalizations about some special cases in division.

- "What do you get when you divide a number by 1?" (You get the number.)
- "What if you divide the number by itself?" (You get 1.)
- "What do you get when you divide zero by a number?" (You get zero.)
- "What if you divide a number by zero?" (That doesn't make sense!)

The last case is the most difficult. Mathematicians call results that don't make sense "undefined".

Dividing multiples of 10

When dividing numbers that end in one or more zeros, is useful to relate the concept of division to multiplication. For example, to divide 3,600 by 90, students can fill the blank in 90×__ = 3,600.

Long Division

We apply the division algorithm from Chapter 8 of Beast Academy 3C to larger numbers.

The goal is that students always understand the steps they are using. Help students reason through division starting with a concrete example.

"If there are 2,868 pennies to split equally into 12 piles, how many pennies will there be in each pile?"

Below is one example of how a student might think through this problem, recording their work as shown on the right. (There are many other ways.)

"I could start by putting 200 pennies in each pile.
That uses a total of 12×200 = 2,400 pennies.
So, it leaves 2,868 − 2,400 = 468 pennies to split up.

Next, I could put 30 more pennies in each pile.
That uses a total of 12×30 = 360 pennies.
So, that leaves 468 − 360 = 108 pennies.

Putting 5 more pennies in each pile leaves 108 − 60 = 48 pennies.

Finally, if I put 4 more pennies in each pile, there will be 0 pennies left over. All together, each pile has a total of 200 + 30 + 5 + 4 = 239 pennies."

Divide 2,868 ÷ 12

4
5
30
200 } 239

12) 2,868
−2,400
468
−360
108
−60
48
−48
0

2,868 ÷ 12 = 239

Why do we use the non-traditional algorithm?

Compare the student reasoning in the previous section to the way a student works through the steps of the traditional algorithm below.

12 goes into 28 2 times. I write a 2 over the 8 and subtract $2\times12=24$ from 28 to get 4.

```
      2
12) 2868
   -24
    4
```

Bring down the 6. Next, I divide 12 into 46.

```
      23
12) 2868
   -24↓
    46
```

12 goes into 46 3 times. I write a 3 over the 6 and subtract $3\times12=36$ from 46 to get 10.

```
      23
12) 2868
   -24
    46
   -36
    10
```

Bring down the 8. Next, I divide 12 into 108.

```
      239
12) 2868
   -24  ↓
    46
   -36
    108
```

12 goes into 108 9 times. I write a 9 on top and subtract $9\times12=108$ from 108 to get 0.

```
      239
12) 2868
   -24
    46
   -36
    108
   -108
      0
```

The algorithm we use has several advantages over to the one above.

The steps make intuitive sense. The math is not hidden. Students are "taking out" equal groups from 2,868, and keeping track of the total. It is clear to students what is happening at each step.

Another advantage is that students can use multiplication facts they are most comfortable with. In each step of the traditional algorithm above, the student must know the *largest* number of times 12 can go into the number they are dividing.

Using our algorithm, students estimate from the beginning and at each step and are less likely to get a nonsensical answer due to a missed digit or other careless error.

Strategies

Students should look for ways to compute division efficiently without using the division algorithm.

For example, to divide by 4, students can halve a number twice. Apply this to a concrete example like cutting a rope in half, then cutting each half in half.

Students can also split a quotient into parts that are easy to divide.
"How could 63,042 marbles be split into 7 buckets equally?"

Students can split 63,000 marbles first to get $63{,}000\div7=9{,}000$ in each bucket. Then, they can split the remaining 42 marbles to get $42\div7=6$ more marbles in each bucket. So, $63{,}042\div7$ is $9{,}000+6=9{,}006$. This strategy will help students understand the divisibility rules that follow.

Divisibility

When we can divide one number by another with no remainder, we say that the first number is divisible by the second.

In this chapter, we help students understand divisibility rules for 2, 5, 10, 100, 4, and 25.

Beast Academy 4
Chapter 6: Logic

Sequence: BA2, Chapter 6 Problem Solving → BA2, Chapter 9 Odds & Evens → BA2, Chapter 12 Problem Solving → BA4, Chapter 6 This Chapter

The strategies in this chapter require very few computational skills. So, the chapter can be used at any time in the series or sprinkled throughout BA level 4.

Overview

Taking given information and using it to arrive at a valid conclusion is **logical thinking**.

The goal is for students to use clues to figure out what they know is definitely true. Sometimes this involves figuring out that something else is definitely false.

Being able to think logically and explain your reasoning is an important skill, particularly in mathematics, which is built on logical reasoning.

While this is a chapter that may be considered optional, **it is a lot of fun**, and we encourage all teachers to include at least some of the puzzles and problems in this chapter.

Teaching Logical Reasoning: Take a Step Back.

None of the puzzles or problems in this chapter require any special math skills. Most of the puzzles are accessible to everyone. Teachers should give students space to figure things out on their own.

Resist the urge to do the work for them!
Instead, encourage students to think with lots of questions.

"What have you tried?"

"Why didn't that work?"

"What did you do next?"

"How do you know?"

"What else can you try?"

This may cause some early frustration. Stay strong! Be patient! Eventually they'll stop asking you for answers and start thinking on their own. Perfect! You're doing it right!

Kids will get a lot more out of these problems if they solve them on their own.

Things you *can* do to help:

- Make sure students understand the puzzles. Do some easy examples together to make sure students know what the rules and goals are.
- If the answer to the question "What have you tried?" is, "I don't remember," encourage students to get organized. Give them ways to keep track of their work. Simple strategies like "draw an X in a box where you know it *can't* be" help students keep track of info.
- Don't be afraid to make mistakes. You can even propose some bad ideas on purpose. This will help students rule stuff out and understand the puzzles better. It also sets the expectation that **students are supposed to experiment and make mistakes**.
- Celebrate variety. Students will find many valid approaches, sometimes ingenius ones! Share these and discuss them, but avoid promoting one "right" way.

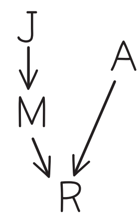
Beast Academy 4
Chapter 6: Logic

Logical Reasoning

“Are you sure?”

Help students recognize the difference between a logical conclusion (something they are sure of) and a hunch (something that just ‘feels’ right). For example, given the facts on the left, which of the statements on the right *must* be true? Which *must* be false? Which ones can’t be determined?

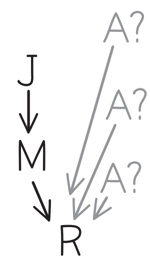
Facts:	**Statements:**
Joey is taller than Mia.	1. Raven is taller than Joey.
Mia is taller than Raven.	2. Raven is the shortest.
Raven is shorter than Andre.	3. Andre is taller than Mia.



We sketch a diagram to organize what we know, drawing arrows from tall to short.

Statement 1: Since Joey is taller than Mia, and Mia is taller than Raven, Joey is definitely taller than Raven. So, statement 1 is false.

Statement 2: We can use the given facts to show that Joey, Mia, and Andre are all taller than Raven. So, Raven is the shortest. Statement 2 is true.



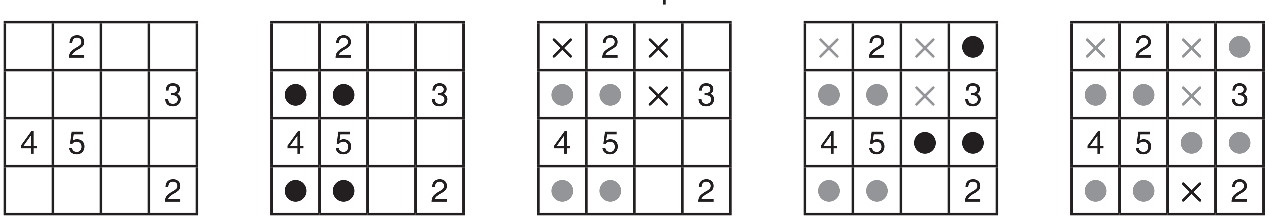
Statement 3: In our first diagram, it looks like Andre is taller than Mia. But are we *sure*? We know that Andre and Mia are both taller than Raven, but we don’t have any information to compare Andre and Mia. We could have drawn Andre *anywhere* above Raven. So, statement 3 can’t be determined.

Students might offer “evidence” like, “A is above M in my diagram,” or just, “Andre seems taller.” Help students learn to support their conclusions using only what they know for sure.

Tipping Over The First Domino

For many of the puzzles in this chapter, the hardest part is getting started. This is where you use your questioning strategies. Once they’ve found a place to start, encourage students to use what they’ve figured out to reach other conclusions. For example, in the Minesweeper puzzle below, each number gives the number of mines that are in the empty squares that surround it. The goal is to figure out where all of the mines are. Encourage to find a godo place to start, then look for what to do next.

Solve the following Minesweeper puzzle.	There is only one way to place 4 mines around the 4.	There can’t be any more mines around the 2 at the top.	Now there is only one way to place 3 mines around the 3.	And the last square can’t have a mine, so we’re done!



“Aha!” moments and chain reactions make problems like this a lot of fun for students and help them learn to reason through all sorts of problems they’ll encounter.

Beast Academy 4
Chapter 7: Factors

Sequence: BA4, Chapter 5 Division → BA4, Chapter 7 This Chapter → BA5, Chapter 5 Factors & Multiples

Students should have their multiplication facts mastered before beginning this chapter. Students should also know divisibility rules from the BA4 division chapter (rules for 2, 4, 5, 10, 25, and 100).

Overview

Factoring is an essential skill that students will use in a variety of situations, from simplifying fractions to working with polynomials in algebra.

Practice with factoring and prime factorization is a great way to improve students' number sense.

Factor Basics

Factors of n are the numbers that n is divisible by. Help students understand the relationship between factors and multiples. For example, factors of 35 are 1, 5, 7, and 35, which also means that 35 is a multiple of 1, 5, 7, and 35.

Encourage students to find factors in an organized way, working in pairs and starting with the smallest. To find all the factors of 40, a student can start with 1×40, then 2×20, then (recognizing that 3 is not a factor) moving on to 4×10 and 5×8.

Next, 6 and 7 are not factors. Since $7\times7=49$, any number that is 7 or greater would have to be paired with a smaller factor. Since we've checked all of the factors smaller than 7, we're done!

List all factors of 40 and of 51.

40:	***51:***
1×40	1×51
2×20	3×17
4×10	
5×8	

Primes and Composites

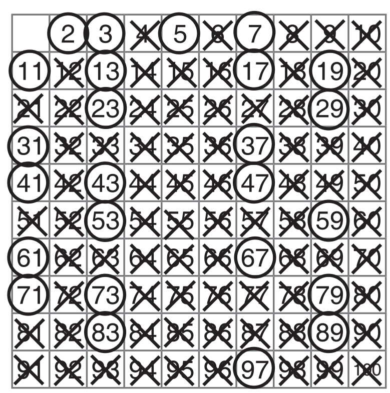
A number with *exactly* two factors (one and itself) is called a **prime** number. A number with more than 2 factors is called **composite**. 0 and 1 are neither prime nor composite.

Students should start to recognize all of the smaller primes (2, 3, 5, 7, and 11). To find every prime under 100, we cross out all of the *composite* numbers from 2 to 100 on a hundreds chart. Every number that's left is prime.

Read through the Gym section of the Guide to help students understand this process. Start by crossing out multiples of 2 (except 2), then multiples of 3 (except 3). Ask students why they don't need to cross out multiples of 4 (those were crossed out when we crossed out the multiples of 2), or 11 (we already crossed out all of the multiples of 11 that are less than 100 since we crossed out every multiple of 2, 3, 4, 5, 6, 7, 8, 9.)

Students can use these charts later to check whether a number is prime or not (51 and 91 are particularly tricky).

In the hundreds chart below, every prime is circled, and every composite is crossed out.



Beast Academy 4
Chapter 7: Factors

Divisibility

Students should be able to recognize whether a number is divisible by 2 (if it's even), 4 (if it ends in a 2-digit multiple of 4), 5 (if it ends in 0 or 5), 10 (if it ends in 0), and possibly other patterns that help them check for divisibility.

We give a few more useful tools for checking divisibility.

Encourage students to split a number into parts. For example, to see whether 42,040 is divisible by 7, students should see that 42,000 is divisible by 7 ($7\times6{,}000$), but 40 is not. So, 42,040 is *not* divisible by 7 (but both 42,035 and 42,042 are).

We also explain divisibility tests for 3 and for 9. The "why" behind these rules is tough, but we encourage you to work through the R&G section of the Guide to help students understand why these rules work. This is an instance where it is acceptable if many students learn the rule without fully understanding it.

Divisibility rule for 3:

If the sum of a number's digits is divisible by 3, then the number is divisible by 3.

For example, since $4+1+8+2=15$, 4,182 is divisible by 3.

Divisibility rule for 9:

If the sum of a number's digits is divisible by 9, then the number is divisible by 9.

For example, since $4+0+9+5=18$, 4,095 is divisible by 9.

Prime Factorization

Every number has its own unique prime factorization. We usually write a number's prime factorization using exponents with the factors listed in order. For example, $54=2\times3^3$, and $60=2^2\times3\times5$. The prime factorization of a prime is the number itself ($71=71$).

Factor trees help students find a number's prime factorization. Discuss (don't discourage) various factor trees for the same number that give the same prime factorization.

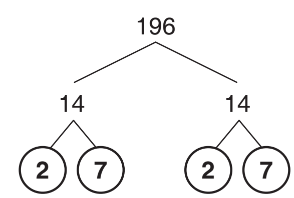
Testing for primes is tricky. How can a student tell whether 127 is prime? Encourage students to be organized and efficient when checking for factors, and know how to tell when they're done.

To see if 127 is prime, we test 2, then 3, then 5. None are factors of 127. Ask, "Why don't we need to test composite numbers like 4 and 6?" If 4 were a factor, 2 would be, too. If 6 were a factor, 2 and 3 would both be factors. **So, we only need to check if primes are factors.** Next, we test 7 and 11. Neither is a factor of 127.

Ask, "Why don't we need to test 13? Or 17? Or 19?" If 13 (or 17, or 19) were a factor of 127, then it would have to be paired with a factor that is smaller than 13, since $13\times13=169$. We already tested all of the primes below 13, and no prime less than 13 is a factor of 127, so 127 is prime!

We can do all sorts of great things with a number's prime factorization. For example, we can tell that 51 (3×17) is a factor of 9,996 ($2^2\times3\times7^2\times17$), since 9,996 has $3\times17=51$ in its prime factorization.

Use a factor tree to find the prime factorization of 196.



$196=2^2\times7^2$.

Is 127 prime?

Divisibility tests tell us that 2, 3, and 5 are not factors of 127. $7\times18=126$, so 7 isn't a factor. $11\times11=121$, so 11 isn't.

So, 127 is prime.

Is $51=3\times17$ a factor of $9{,}996=2^2\times3\times7^2\times17$?

$$9{,}996=(3\times17)\times(2^2\times7^2)$$
$$=51\times(2^2\times7^2)$$

So, 51 is a factor of 9,996.

Beast Academy 4
Chapter 8: Fractions (+&−)

AoPS Beast Academy
Art of Problem Solving

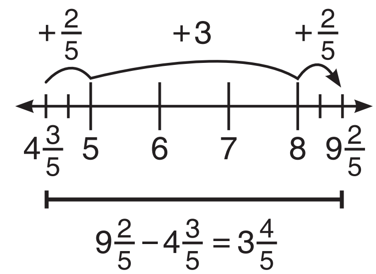
Sequence: BA3, Chapter 10 Fractions Basics → **BA4, Chapter 8 This Chapter** → BA4, Chapter 10 Fractions ×&÷ → BA5, Chapter 6 Fraction Computations

Students **must** have a solid understanding of fractions and fraction notation from BA3, Chapter 10 before beginning this chapter.

Overview

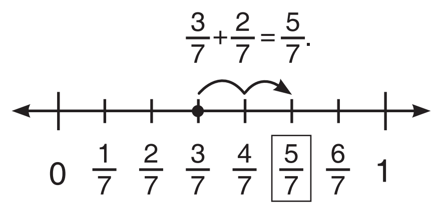
This chapter introduces adding and subtracting fractions and mixed numbers with the same denominator.

Focus on fractions as **numbers on the number line**. When students think of fractions as numbers (and not just as parts of a whole), they can apply strategies they've use with whole numbers, like the counting-up strategy for subtraction shown on the right.



Adding and Subtracting Fractions

Relate fraction addition and subtraction to the computations students already know. Adding 3 pencils and 2 pencils equals 5 pencils. Adding 3 sevenths plus 2 sevenths equals 5 sevenths.



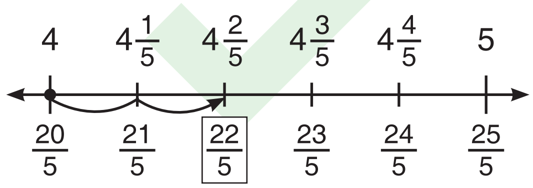
Mixed Number Conversions

Avoid teaching an algorithmic approach to conversions. Instead, encourage students to reason through conversions using what they know about fractions.

Encourage This

$4\frac{2}{5}$ is $4+\frac{2}{5}$, and 4 is $\frac{20}{5}$.

So, $4\frac{2}{5}=\frac{20}{5}+\frac{2}{5}=\frac{22}{5}$.



Similarly, $\frac{22}{5}=\frac{20}{5}+\frac{2}{5}=4+\frac{2}{5}=4\frac{2}{5}$.

Not This

$4\frac{2}{5}$

1. Multiply the whole number by the denominator. — 1. $4\times5=20$
2. Add the numerator to this result. — 2. $20+2=22$
3. Put your final result over the original denominator. — 3. $\frac{22}{5}$

Beast Academy 4
Chapter 8: Fractions

Breaking and Regrouping

Students should understand that they can **break** the whole number part of a mixed number to make a subtraction problem easier.

$$3\tfrac{1}{5} = 2 + \tfrac{5}{5} + \tfrac{1}{5} = 2\tfrac{6}{5}$$

Students should also be able to do the opposite and **regroup** the fraction part of a mixed number to write final answers.

$$2\tfrac{6}{5} = 2 + \tfrac{5}{5} + \tfrac{1}{5} = 3\tfrac{1}{5}$$

Adding and Subtracting Mixed Numbers

For addition, encourage students to add the whole numbers and fractions separately, then regroup. For subtraction, encourage students to break a whole into parts to make the subtraction easier. This is almost always better than converting mixed numbers to fractions before adding or subtracting.

Encourage This

$$\begin{aligned}6\tfrac{5}{7} + 2\tfrac{4}{7} &= \left(6 + \tfrac{5}{7}\right) + \left(2 + \tfrac{4}{7}\right)\\ &= (6+2) + \left(\tfrac{5}{7} + \tfrac{4}{7}\right)\\ &= 8 + \tfrac{9}{7}\\ &= 9\tfrac{2}{7}.\end{aligned}$$

$$\begin{array}{r} 6\tfrac{4}{7} \\ -2\tfrac{6}{7} \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{r} 5\tfrac{11}{7} \\ -2\tfrac{6}{7} \\ \hline 3\tfrac{5}{7} \end{array}$$

We do not expect students to show work this way. This is the thought process we'd like students to be using. Many steps can be done in their heads.

Not This

$$\begin{aligned}6\tfrac{5}{7} + 2\tfrac{4}{7} &= \left(\tfrac{6\times7+5}{7}\right) + \left(\tfrac{2\times7+4}{7}\right)\\ &= \tfrac{47}{7} + \tfrac{18}{7}\\ &= \tfrac{65}{7}\\ &= 9\tfrac{2}{7}.\end{aligned}$$

$$\begin{aligned}6\tfrac{4}{7} - 2\tfrac{6}{7} &= \left(\tfrac{6\times7+4}{7}\right) - \left(\tfrac{2\times7+6}{7}\right)\\ &= \tfrac{46}{7} - \tfrac{20}{7}\\ &= \tfrac{26}{7}\\ &= 3\tfrac{5}{7}.\end{aligned}$$

Computation Strategies

We can use the same strategies we use with whole numbers when adding and subtracting mixed numbers and fractions. Examples are given below.

Clever Regrouping.

$$\begin{aligned}10\tfrac{8}{9} + \tfrac{5}{9} &= 10\tfrac{8}{9} + \left(\tfrac{1}{9} + \tfrac{4}{9}\right)\\ &= \left(10\tfrac{8}{9} + \tfrac{1}{9}\right) + \tfrac{4}{9}\\ &= 11\tfrac{4}{9}\end{aligned}$$

Subtract then add.

$$\begin{aligned}4\tfrac{1}{9} - 2\tfrac{8}{9} &= 4\tfrac{1}{9} - 3 + \tfrac{1}{9}\\ &= 1\tfrac{1}{9} + \tfrac{1}{9}\\ &= 1\tfrac{2}{9}.\end{aligned}$$

Make easy sums.

$$\begin{aligned}\tfrac{1}{7} + \tfrac{3}{7} + \tfrac{4}{7} + \tfrac{6}{7} &= \left(\tfrac{1}{7} + \tfrac{6}{7}\right) + \left(\tfrac{3}{7} + \tfrac{4}{7}\right)\\ &= 1 + 1\\ &= 2.\end{aligned}$$

UNDER CONSTRUCTION

(We're still working on this one.)

UNDER CONSTRUCTION

(We're still working on this one.)

Beast Academy 4
Chapter 11: Decimals

AoPS Beast Academy
Art of Problem Solving

Sequence: BA4, Chapter 11 This Chapter ▶ BA5, Chapter 9 Decimals

This is the first time we introduce decimals in Beast Academy. Students should already be familiar with fractions.

Overview

Decimals are just another way to write fractions. We emphasize a place value understanding of decimals. Everything students already know about numbers can be extended to these smaller place values. Avoid tricks that hide a place value understanding of decimals.

Place Value

Place value is key to understanding how decimals and fractions relate.

In our number system, each place value is ten times the place value to its right (and ten times smaller than the place value to its left). To the right of the 1's place, we have the $1 \div 10 = \frac{1}{10}$'s (tenths) place, followed by the hundredths place, thousandths place, and so on.

The decimal point is always between the ones place and the tenths place.

2	3	4	.	5	6	7	8
100's	10's	1's		$\frac{1}{10}$'s	$\frac{1}{100}$'s	$\frac{1}{1,000}$'s	$\frac{1}{10,000}$'s

Converting Between Decimals and Fractions

Encourage students to use what they know about place value and equivalent fractions to convert between fractions and decimals.

For example, to write $\frac{453}{1,000}$ as a decimal, we can split it into parts, similar to the way we split whole numbers into ones, tens, and hundreds.

$$\frac{453}{1,000} = \frac{400}{1,000} + \frac{50}{1,000} + \frac{3}{1,000}$$
$$= \frac{4}{10} + \frac{5}{100} + \frac{3}{1,000}$$

So, $\frac{453}{1,000}$ is 4 tenths, 5 hundredths, and 3 thousandths, which we write in decimal form as 0.453.

We can reverse this to convert 0.453 to $\frac{453}{1,000}$.

With practice, students should see that any 3-digit number after the decimal point is a number of thousandths.

Avoid This

1. To convert 0.453 to a fraction, we count the digits right of the decimal point (3). — 1. 0.453 (3)
2. Write this number of zeros after a 1 in the bottom of the fraction. — 2. $\frac{}{1,000}$
3. Write the number in the decimal on top. — 3. $\frac{453}{1,000}$

Beast Academy 4
Chapter 11: Decimals

Comparing and Ordering

Again, focus on place value. Students who rely on patterns they used to compare whole numbers may make some common mistakes.

Where the digits are is more important than what the digits are. Even though 4 is larger than 3, students should recognize that 0.04 is less than 0.3. They can write both as fractions and compare, but should eventually recognize that digits in the larger place value always outweigh those in smaller place values.

Which is greater, 0.04 or 0.3?

$$0.04 = \frac{4}{100}$$

$$0.3 = \frac{3}{10} = \frac{30}{100}$$

$$\frac{4}{100} < \frac{30}{100}, \text{ so } 0.04 < 0.3.$$

Having more digits does not always make a number larger. Guide students to understand why 7.8 is greater than 7.77 with questions like, "Which decimal has more ones? More tenths? Can you write each fraction as a number of hundredths?"

Guide them to understand trailing zeros with questions like, "Which is greater, 0.9 or 0.90?" Trailing 0's don't change a decimal's value.

Which is greater, 7.8 or 7.77?

7.8 and 7.77 have the same number of ones, but 7.8 has more tenths, so 7.<u>8</u> > 7.<u>7</u>7.

Align by place value. It's easiest to order decimals if their decimal points and all of their place values are aligned. This way, we can compare the digits in the largest place values first.

It may help students to add trailing zeros to make comparing decimals similar to comparing whole numbers. But the goal is for students to compare decimals without adding trailing zeros.

Order 0.2, 0.22, 0.202, and 0.022 from greatest to least.

greatest
0.220
0.202
0.200
0.022
least

Addition and Subtraction

Once again, focus on place value. Adding and subtracting decimals is no different from adding and subtracting whole numbers. Align place values and add or subtract the smaller place values first, regrouping or breaking as necessary.

Encourage students to apply the same **mental strategies** they have used with whole numbers to decimal addition and subtraction.

Compute 9.53 + 6.7 and 9.6 − 5.83.

$$\begin{array}{r} \scriptsize{1} \\ 9.53 \\ +\ 6.70 \\ \hline 16.23 \end{array} \qquad \begin{array}{r} \scriptsize{8\ \ 15\ 10} \\ \not{9}.\not{6}\not{0} \\ -\ 5.83 \\ \hline 3.77 \end{array}$$

For example, to add 0.75 + 3.4 + 1.25, we first add 0.75 + 1.25 to get 2, then add 2 + 3.4 to get 5.4.

To subtract 2.98 from 8.92, we can subtract 3, then add back 0.02.

Compute 0.75 + 3.4 + 1.25.

$$\begin{aligned} &0.75+3.4+1.25 \\ &= (0.75+1.25)+3.4 \\ &= 2+3.4 \\ &= 5.4 \end{aligned}$$

Compute 8.92 − 2.98.

$$\begin{aligned} &8.92-2.98 \\ &= 8.92-3+0.02 \\ &= 5.92+0.02 \\ &= 5.94 \end{aligned}$$

UNDER CONSTRUCTION

(We're still working on this one.)